Optimal covariance matrix estimators in a seemingly unrelated regression model

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Abstract
A seemingly unrelated regression model is a set of multiple regression models with correlations. Recently, some aspects of optimal equivariant estimation of the regression vectors and the covariance matrix in a seemingly unrelated regression model have been discussed by the authors. As for the covariance matrix estimation, the best equivariant estimator under Stein's loss function was derived. In this paper, other loss functions are considered, which leads to different best equivariant estimators of the covariance matrix. The differences between the estimators are also examined.

Keywords
Best equivariant estimator; Covariance matrix; Error variance; Multiple regression model; Seemingly unrelated regression model

1. Introduction

A seemingly unrelated regression (SUR) model is a set of regression models

$$\boldsymbol{y}_i = \boldsymbol{X}_i\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i \quad \left(E[\boldsymbol{\varepsilon}_i] = \boldsymbol{0}_n,\ V[\boldsymbol{\varepsilon}_i] = \sigma_i^2\boldsymbol{I}_n\right),\quad i = 1, \dots, p$$

with correlations

$$E\left[\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_j'\right] = \sigma_i\sigma_j r_{ij}\boldsymbol{I}_n,\quad i, j = 1, \dots, p\ (i \neq j),$$

where $\boldsymbol{y}_i$ is an n-dimensional response vector, $\boldsymbol{X}_i$ is an $n \times k_i$ explanatory matrix of rank k_i, $\boldsymbol{\beta}_i$ is a k_i-dimensional regression vector, $\boldsymbol{\varepsilon}_i$ is an n-dimensional error vector, $\boldsymbol{0}_n$ is the n-dimensional zero vector, $\boldsymbol{I}_n$ is the $n \times n$ identity matrix, $\sigma_i^2 (> 0)$ is the error variance of the ith regression model, and r_{ij} is the correlation coefficient between the error terms of the ith and jth regression models. Let

$$\boldsymbol{y} = \left(\boldsymbol{y}_1', \dots, \boldsymbol{y}_p'\right)',\ \ \boldsymbol{X} = \mathrm{diag}\{\boldsymbol{X}_1, \dots, \boldsymbol{X}_p\},\ \ \boldsymbol{\beta} = \left(\boldsymbol{\beta}_1', \dots, \boldsymbol{\beta}_p'\right)',\ \ \boldsymbol{\varepsilon} = \left(\boldsymbol{\varepsilon}_1', \dots, \boldsymbol{\varepsilon}_p'\right)',$$
$$\boldsymbol{\Sigma}_d = \mathrm{diag}\{\sigma_1, \dots, \sigma_p\},\ \ r_{ii} = 1\ (i = 1, \dots, p),\ \ \boldsymbol{\Lambda} = \left(r_{ij}\right)_{i,j=1,\dots,p},$$

where $\boldsymbol{\Lambda}$ is assumed to be full rank and thus positive definite. Then, we have

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \quad \left(E[\boldsymbol{\varepsilon}] = \boldsymbol{0}_{np},\quad V[\boldsymbol{\varepsilon}] = \boldsymbol{\Sigma} \otimes \boldsymbol{I}_n = \boldsymbol{\Sigma}_d\boldsymbol{\Lambda}\boldsymbol{\Sigma}_d \otimes \boldsymbol{I}_n\right),$$

where $\otimes$ denotes the Kronecker product.

The SUR model has been widely discussed and applied to various areas (Zellner (1962), Srivastava and Giles (1987), Kariya and Kurata (2004), Shah et al. (2004)). Although this paper focuses on equivariant estimation in SUR model, recent studies on various aspects of SUR model include, for example, Alkhamisi et al. (2019), Gong (2019), Jiang et al. (2020), and Zhao et al. (2018).

2. Best equivariant estimators in SUR model

Suppose that the probability density function of $\boldsymbol{\varepsilon}$ is elliptically symmetric:

$$g(\boldsymbol{\varepsilon}) = |\boldsymbol{\Sigma}|^{-\frac{n}{2}} h\left(\boldsymbol{\varepsilon}'(\boldsymbol{\Sigma} \otimes \boldsymbol{I}_n)^{-1}\boldsymbol{\varepsilon}\right) \text{ for some known probability density function } h$$

and the correlation matrix $\boldsymbol{\Lambda}$ is known. Kurata and Matsuura (2016) derived the best equivariant estimator of the integrated regression vector $\boldsymbol{\beta}$ under the following loss function

$$L(\widehat{\boldsymbol{\beta}}) = (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \boldsymbol{X}' (\boldsymbol{\Sigma} \otimes \boldsymbol{I}_n)^{-1} \boldsymbol{X} (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}),$$

and Matsuura and Kurata (2020) gave the best equivariant estimator of the covariance matrix $\boldsymbol{\Sigma}$ under the Stein's loss function

$$L_1(\widehat{\boldsymbol{\Sigma}}) = \mathrm{tr}\{\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}\} - \log|\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}| - p.$$

Here, estimators $\widehat{\boldsymbol{\beta}}_i(\boldsymbol{y}_1, \dots, \boldsymbol{y}_p), i = 1, \dots, p$ and $\widehat{\boldsymbol{\Sigma}}(\boldsymbol{y}_1, \dots, \boldsymbol{y}_p)$ are called equivariant if

$$\widehat{\boldsymbol{\beta}}_i(a_1\boldsymbol{y}_1 + \boldsymbol{X}_1\boldsymbol{c}_1, \dots, a_p\boldsymbol{y}_p + \boldsymbol{X}_p\boldsymbol{c}_p) = a_i\widehat{\boldsymbol{\beta}}_i(\boldsymbol{y}_1, \dots, \boldsymbol{y}_p) + \boldsymbol{c}_i, \ \ i = 1, \dots, p,$$

$$\widehat{\boldsymbol{\Sigma}}(a_1\boldsymbol{y}_1 + \boldsymbol{X}_1\boldsymbol{c}_1, \dots, a_p\boldsymbol{y}_p + \boldsymbol{X}_p\boldsymbol{c}_p) = \boldsymbol{A}\widehat{\boldsymbol{\Sigma}}(\boldsymbol{y}_1, \dots, \boldsymbol{y}_p)\boldsymbol{A} \text{ with } \boldsymbol{A} = \mathrm{diag}\{a_1, \dots, a_p\}$$

for any $a_i > 0$, $\boldsymbol{c}_i \in R^{k_i}$, $i = 1, \dots, p$.

Let $\boldsymbol{e}_i = (\boldsymbol{I}_n - \boldsymbol{X}_i(\boldsymbol{X}_i'\boldsymbol{X}_i)^{-1}\boldsymbol{X}_i')\boldsymbol{y}_i$, $\boldsymbol{u}_i = \frac{\boldsymbol{e}_i}{\|\boldsymbol{e}_i\|}$, $i = 1, \dots, p$, $\boldsymbol{K} = \mathrm{diag}\{\|\boldsymbol{e}_1\|, \dots, \|\boldsymbol{e}_p\|\}$, and $\boldsymbol{u} = (\boldsymbol{u}_1', \dots, \boldsymbol{u}_p')'$. Matsuura and Kurata (2020) showed that the best equivariant estimator of $\boldsymbol{\Sigma}$ under the Stein's loss function L_1 is given by

$$\widehat{\boldsymbol{\Sigma}}_1 = \boldsymbol{K}(\boldsymbol{T}(\boldsymbol{u}) \circ \boldsymbol{\Lambda}^{-1})^{-1}\boldsymbol{K}$$

$$\text{with } t_{ij}(\boldsymbol{u}) = E_{(\boldsymbol{0}_k, \Lambda)}[\|\boldsymbol{e}_i\|\|\boldsymbol{e}_j\||\boldsymbol{u}], \ \ i, j = 1, \dots, p, \ \ \boldsymbol{T}(\boldsymbol{u}) = (t_{ij}(\boldsymbol{u}))_{i,j=1,\dots,p},$$

where $\circ$ denotes the Hadamard product and $E_{(\boldsymbol{0}_k, \Lambda)}[\ \ ]$ denotes the expectation under $\boldsymbol{\beta} = \boldsymbol{0}_k$ ($k = \sum_{i=1}^p k_i$) and $\boldsymbol{\Sigma} = \boldsymbol{\Lambda}$. Matsuura and Kurata (2020) also showed that the best equivariant estimator of $\boldsymbol{\beta}$ given by Kurata and Matsuura (2016) is expressed as the generalized least squares estimator using $\widehat{\boldsymbol{\Sigma}}_1$, that is, $\widehat{\boldsymbol{\beta}} = \left(\boldsymbol{X}'(\widehat{\boldsymbol{\Sigma}}_1 \otimes \boldsymbol{I}_n)^{-1}\boldsymbol{X}\right)^{-1} \boldsymbol{X}'(\widehat{\boldsymbol{\Sigma}}_1 \otimes \boldsymbol{I}_n)^{-1}\boldsymbol{y}$.

3. Best equivariant estimators of the covariance matrix under other loss functions in SUR model

This paper considers other loss functions for the estimation of $\boldsymbol{\Sigma}$. More specifically, we present the best equivariant estimators of $\boldsymbol{\Sigma}$ under the following loss functions:

$$L_2(\widehat{\boldsymbol{\Sigma}}) = \mathrm{tr}\left\{(\widehat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1} - \boldsymbol{I}_p)^2\right\},$$

$$L_3(\widehat{\boldsymbol{\Sigma}}) = \mathrm{tr}\{\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\} - \log|\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}| - p.$$

The loss function $L_2(\widehat{\boldsymbol{\Sigma}})$ is a common loss function for the covariance matrix estimation as well as the Stein's loss function $L_1(\widehat{\boldsymbol{\Sigma}})$. The loss function $L_3(\widehat{\boldsymbol{\Sigma}})$ can be viewed as the Stein's loss function for the estimation of the precision matrix $\boldsymbol{\Sigma}^{-1}$.

Proposition 1. Let ρ_{ij} denote the (i, j)th element of $\boldsymbol{\Lambda}^{-1}$. Let also

$$\boldsymbol{\rho}_j = (\rho_{1j}, \dots, \rho_{pj})', \ \ j = 1, \dots, p, \ \ \boldsymbol{\rho} = \mathrm{vec}(\boldsymbol{\Lambda}^{-1}) = (\boldsymbol{\rho}_1', \dots, \boldsymbol{\rho}_p')', \ \ \boldsymbol{\Pi} = (\boldsymbol{\rho}_j\boldsymbol{\rho}_i')_{i,j=1,\dots,p},$$

$$\boldsymbol{k} = (\|\boldsymbol{e}_1\|, \dots, \|\boldsymbol{e}_p\|)', \ \ \boldsymbol{m}_i(\boldsymbol{u}) = E_{(\boldsymbol{0}_k, \Lambda)}[\|\boldsymbol{e}_i\|\boldsymbol{k}|\boldsymbol{u}], i = 1, \dots, p, \ \ \boldsymbol{m}(\boldsymbol{u}) = (\boldsymbol{m}_1(\boldsymbol{u})', \dots, \boldsymbol{m}_p(\boldsymbol{u})')',$$

$$\boldsymbol{M}_{ij}(\boldsymbol{u}) = E_{(\boldsymbol{0}_k, \Lambda)}[\|\boldsymbol{e}_i\|\|\boldsymbol{e}_j\|\boldsymbol{k}\boldsymbol{k}'|\boldsymbol{u}], i, j = 1, \dots, p, \ \ \boldsymbol{M}(\boldsymbol{u}) = (\boldsymbol{M}_{ij}(\boldsymbol{u}))_{i,j=1,\dots,p}.$$

Suppose that $\boldsymbol{M}(\boldsymbol{u}) \circ \boldsymbol{\Pi}$ is invertible. Then, the best equivariant estimator of $\boldsymbol{\Sigma}$ under the loss function L_2 is given by

$$\widehat{\boldsymbol{\Sigma}}_2 = \boldsymbol{K}\boldsymbol{Q}(\boldsymbol{u})\boldsymbol{K} \text{ with } \mathrm{vec}(\boldsymbol{Q}(\boldsymbol{u})) = (\boldsymbol{M}(\boldsymbol{u}) \circ \boldsymbol{\Pi})^{-1}(\boldsymbol{m}(\boldsymbol{u}) \circ \boldsymbol{\rho}).$$

Proposition 2. Suppose that $E_{(\boldsymbol{0}_k, \Lambda)}\left[\frac{1}{\|\boldsymbol{e}_i\|\|\boldsymbol{e}_j\|}\right], i, j = 1, \dots, p$ are finite. Let

$$s_{ij}(\boldsymbol{u}) = E_{(\boldsymbol{0}_k, \Lambda)}\left[\frac{1}{\|\boldsymbol{e}_i\|\|\boldsymbol{e}_j\|}\middle|\boldsymbol{u}\right], \ \ i, j = 1, \dots, p, \ \ \boldsymbol{S}(\boldsymbol{u}) = (s_{ij}(\boldsymbol{u}))_{i,j=1,\dots,p}.$$

Then, the best equivariant estimator of $\boldsymbol{\Sigma}$ under the loss function L_3 is given by

$$\widehat{\boldsymbol{\Sigma}}_3 = \boldsymbol{K}(\boldsymbol{S}(\boldsymbol{u}) \circ \boldsymbol{\Lambda})\boldsymbol{K}.$$

Proofs of the propositions are omitted in this paper. These results indicate that different loss functions L_1, L_2, and L_3 lead to different best equivariant estimators $\widehat{\boldsymbol{\Sigma}}_1$, $\widehat{\boldsymbol{\Sigma}}_2$, and $\widehat{\boldsymbol{\Sigma}}_3$. In particular, $\widehat{\boldsymbol{\Sigma}}_1$, $\widehat{\boldsymbol{\Sigma}}_2$, and $\widehat{\boldsymbol{\Sigma}}_3$ use different moments of $\|\boldsymbol{e}_i\|, i = 1, \dots, p$ conditioned on $\boldsymbol{u}$. As noted in Section 2, the best equivariant estimator of $\boldsymbol{\beta}$ given by Kurata and Matsuura (2016) under the loss function $L(\hat{\boldsymbol{\beta}})$ is expressed as the generalized least squares estimator using $\hat{\boldsymbol{\Sigma}}_1$. It might be expected that the generalized least squares estimators using $\hat{\boldsymbol{\Sigma}}_2$ and $\hat{\boldsymbol{\Sigma}}_3$ may also be the best equivariant estimators of $\boldsymbol{\beta}$ under other loss functions, which will be a topic for future research.

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