On the study of extinction of circuit chains describing random walks in random environments

Ganatsiou Chrysoula
Department of Digital Systems, University of Thessaly,
Gaiopolis Campus, Larissa-Trikala Ring Road, 41500 Larissa, Greece.
e-mail: chganats@uth.gr, ganatsio@otenet.gr

Abstract. By considering a nonhomogeneous random walk with jumps (with steps -1 or +1 or in the same position having a right elastic barrier at 0) we investigate appropriate expressions of the mean time to extinction for the corresponding Markov chain describing uniquely the abovementioned random walk through its unique representation by directed circuits and weights (circuit chain) in random environments.
Keywords: *circuit representation theory, random walk with jumps, random environment, circuit chain, mean first passage time, extinction time.*

1. Introduction. It is known that *extinction* means the termination of a kind of organism or of a system. The moment of extinction is generally the time when the system entries the absorbing state 0 although the capacity to recover may have been lost before this point. Since the states where the system entries before the extinction may be too many the determination of this moment is difficult and it is usually done retrospectively ([6], [11]). A central question in this concept is the probability of ultimate extinction where no system exists after some finite number of transitions. To this direction the determination of a measure known as the "*expected time to extinction*" or "*mean survival time*" which possesses logical properties is very fundamental. The expected time to extinction of a system is intimately related to the probability of its occurrence in such a way so as to ensure the validity of certain common inference patterns found optimization, adaptationist and similar types of evolutionary reasoning ([10]).

In parallel a systematic work has been developed (Kalpazidou [7], MacQueen [8], Qian Minping and Qian Min [9], Zemanian [12] and others) in order to investigate representations of the finite-dimensional distributions of Markov processes (with discrete or continuous parameter) having an invariant measure, as decompositions in terms of the *circuit passage functions*

$$J_c(i,j)=\begin{cases}1, & \text{if } \ i,j \ \text{ are consecutive states of } c,\\ 0, & \text{otherwise,}\end{cases}$$

for any directed sequence c= $(i_1, i_2, \ldots, i_v, i_1)$ of states called a *circuit*, v>1. The representations are called *circuit representations* while the corresponding discrete parameter Markov processes generated by directed circuits c = $(i_1,i_2,\ldots, i_v,i_1)$, v>1, are called *circuit chains*.

Following the context of the theory of Markov processes′ circuit representation (Kalpazidou [7]), the present works arises as an attempt to investigate suitable expressions of the mean first passage time to the state 0 starting at state k, $k\in Z$, of the corresponding Markov chain (circuit chain) describing uniquely by directed circuits and weights a nonhomogeneous random walk with jumps (with steps − 1 or + 1 or in the same position having a right elastic barrier at 0) in random environments. Since for any such Markov chain with an absorbing state the extinction usually occurs as the time $t\to\infty$ with unit probability this will give us a new perspective in the whole study (For an analogous study of the discrete-time birth-death circuit chains as a special case you may see Ganatsiou [3, 4]).

2. Markov chain's circuit representation of a random walk with jumps in random environments. Let us consider the random walk on Z, with transitions $k\to(k-1)$, $k\to(k+1)$ and $k\to k$, whose transition probabilities $(p_k)_{k\in Z}$, $(r_k)_{k\in Z}$ constitute stationary ergodic sequences. A realization of these sequences is called a *random environment*. In order to investigate the unique circuit and weight representation of this random walk in random environments, for almost every environment, let us consider a probability space (Ω, F, μ), a measure preserving ergodic automorphism of this space m: Ω →Ω and the measurable functions p: Ω → (0, 1), r: Ω → (0, 1) such that every $\omega \in \Omega$ generates the random environment $p_k \equiv p(m^k\omega)$, $r_k \equiv r(m^k\omega)$, k∈Z. Since m is measure preserving and ergodic, the sequences $(p_k)_{k\in Z}$, $(r_k)_{k\in Z}$ are stationary ergodic sequences of random variables. Let also S = Z^N be the infinite product space with coordinates $(X_n)_{n\in N}$. Then we may define a family $(P^{\omega})_{\omega\in\Omega}$ of probability measures such that, for every ω∈Ω, the sequence $(X_n)_{n\in N}$ forms a Markov chain on Z whose the elements of the corresponding Markov transition matrix are defined by

$P^{\omega}(X_0 = 0) = 1,$

$P^{\omega}(X_{n+1} = k+1 / X_n = k) = p(m^k\omega),$

$P^{\omega}(X_{n+1} = k / X_n = k) = r(m^k\omega),$

$P^{\omega}(X_{n+1} = k-1 / X_n = k) = 1 - p(m^k\omega) - r(m^k\omega) \equiv q(m^k\omega), k \in Z,$

as it is shown in Figure 1.

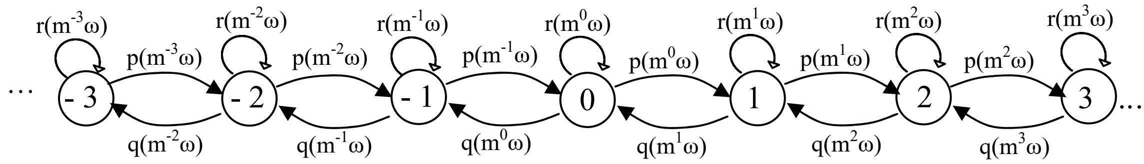

Figure 1

By considering the set of directed circuits c_k = (k, k+1, k) and c'_k = (k, k), for every k∈Z, and the sequences $(b_k(\omega))_{k\in Z}$, $(\gamma_k(\omega))_{k\in Z}$ defined by

$$b_k(\omega) = \frac{w_k(\omega)}{w_{k-1}(\omega)}, \quad \gamma_k(\omega) = \frac{w'_k(\omega)}{w'_{k-1}(\omega)}, \quad \text{for every } k \in Z$$

or equivalently by

$$b_k(\omega) = \frac{p(m^k\omega)}{1 - p(m^k\omega) - r(m^k\omega)} = \frac{p(m^k\omega)}{q(m^k\omega)},$$

$$\gamma_k(\omega) = \frac{r(m^k\omega)}{r(m^{k-1}\omega)} \cdot \frac{p(m^{k-1}\omega)}{p(m^k\omega)} \cdot b_k(\omega), \quad \text{for every } k \in Z$$

we may obtain that given the stationary ergodic sequences $(p_k)_{k\in Z}$, $(r_k)_{k\in Z}$, for which every $\omega \in \Omega$ generates the random environment $p_k \equiv p(m^k\omega)$, $r_k \equiv r(m^k\omega)$, $k \in Z$, the preceding equations define uniquely the sequences $(b_k(\omega))_{k\in Z}, (\gamma_k(\omega))_{k\in Z}$, for μ-almost every ω. Consequently the sequences of weights $(w_k(\omega))_{k\in Z}$ and $(w'_k(\omega))_{k\in Z}$ are defined uniquely by

$$w_k(\omega) = w_0(\omega) \cdot b_1(\omega) \cdot b_2(\omega) \cdots b_k(\omega), \quad k \in Z_+^*,$$

$$w_k(\omega) = \frac{w_0(\omega)}{b_0(\omega) \cdot b_{-1}(\omega) \cdot b_{-2}(\omega) \cdots b_{k+1}(\omega)}, \quad k \in Z_-^*$$

and

$$w'_k(\omega) = w'_0(\omega) \cdot \gamma_1(\omega) \cdot \gamma_2(\omega) \cdots \gamma_k(\omega), \quad k \in Z_+^*,$$

$$w'_k(\omega) = \frac{w'_0(\omega)}{\gamma_0(\omega) \cdot \gamma_{-1}(\omega) \cdot \gamma_{-2}(\omega) \cdots \gamma_{k+1}(\omega)}, \quad k \in Z_-^*,$$

(the unicity of the weight sequences $(w_k(\omega))_{k\in Z}$, $(w'_k(\omega))_{k\in Z}$ is understood up to the constant factors $w_0(\omega)$, $w'_0(\omega)$).

Therefore we have the following (Ganatsiou [1, 2, 5])

Proposition 1. *The Markov chain $(X_n)_{n\in N}$ has a unique circuit and weight representation for μ almost every environment $\omega \in \Omega$.*

3. Extinction time of the circuit chain associated with a random walk with jumps in random environments.

By using the above conclusions and by distinguishing two cases: (i) $\ell = 2, 3, ...,$ (ii) $\ell = ..., -3, -2$ the present study managed to investigate appropriate approximations of the mean time to extinction $t_\ell(\omega)$, $\omega \in \Omega$.

To this direction for the case $\ell \in Z_+^* \setminus \{1\}$ by considering that the state 0 is a recurrent, absorbing state we may take that the expected time before the random walker entries in the state 0 conditioned that the initial state is the state ℓ, $\ell \in Z_+^* \setminus \{1\}$, $\omega \in \Omega$, is given by the relation

$$t_\ell(\omega) = t_1(\omega) + \sum_{k=1}^{\ell-1} \frac{q(m^1\omega)...q(m^k\omega)}{p(m^1\omega)...p(m^k\omega)}\left[t_1(\omega) - \frac{1}{q(m^1\omega)} - \sum_{i=2}^{k} \frac{p(m^1\omega)...p(m^{i-1}\omega)}{q(m^1\omega)...q(m^i\omega)}\right], \quad \ell = 2, 3, ..., \quad \omega \in \Omega .$$

In order to determine the exact value of $t_1(\omega)$, $\omega \in \Omega$, we consider the stationary distribution $\pi_\ell(\omega)$, $\omega \in \Omega$, $\ell \in Z_+^* \setminus \{1\}$, of the modified nonhomogeneous random walk with jumps ($p(m^0\omega) = 1$) in random environments described by the circuit chain $(X_n)_{n\in N}$. Taking into account that $t_1(\omega) = \frac{1}{\pi_0(\omega)} - 1$, we determine $\pi_0(\omega)$ by the relation

$$\pi_0(\omega) = \left[1 + \sum_{k=1}^{+\infty} \frac{p(m^1\omega)...p(m^{k-1}\omega)}{q(m^1\omega)...q(m^k\omega)}\right]^{-1}$$

if and only if $\sum_{k=1}^{+\infty} \frac{p(m^1\omega)...p(m^{k-1}\omega)}{q(m^1\omega)...q(m^k\omega)} < +\infty$, $\omega \in \Omega$.

Furthermore since

$$b_1(\omega) \cdot b_2(\omega)...b_k(\omega) = \frac{p(m^1\omega)...p(m^k\omega)}{q(m^1\omega)...q(m^k\omega)} = \frac{w_k(\omega)}{w_0(\omega)}, \quad k \in Z_+^*, \omega \in \Omega,$$

$$\gamma_1(\omega) \cdot \gamma_2(\omega)...\gamma_k(\omega) = \frac{p(m^0\omega)}{r(m^0\omega)} \cdot \frac{r(m^k\omega)}{p(m^k\omega)} \cdot [b_1(\omega) \cdot b_2(\omega)...b_k(\omega)] = \frac{w'_k(\omega)}{w'_0(\omega)}, \quad k \in Z_+^*, \omega \in \Omega,$$

we may obtain that

$$t_\ell(\omega) = t_1(\omega) + \sum_{k=1}^{\ell-1}\left[\frac{1 - p(m^k\omega) - q(m^k\omega)}{w_k(\omega) \cdot q(m^k\omega)} \cdot \frac{w_{k-1}(\omega)}{w'_k(\omega)} \cdot \sum_{i=k+1}^{+\infty} \frac{w'_i(\omega)}{1 - p(m^i\omega) - q(m^i\omega)}\right], \quad \ell = 2, 3, ..., \quad \omega \in \Omega . \tag{3.1}$$

Following an analogous way for $\ell \in Z_-^* \setminus \{-1\}$ we may take that

$$t_\ell(\omega) = t_{-1}(\omega) + \sum_{k=-1}^{\ell+1}\left[\frac{1 - p(m^k\omega) - q(m^k\omega)}{p(m^k\omega) \cdot w_{k-1}(\omega)} \cdot \frac{w_k(\omega)}{w'_k(\omega)} \cdot \sum_{i=k-1}^{-\infty} \frac{w'_i(\omega)}{1 - p(m^i\omega) - q(m^i\omega)}\right], \quad \ell = ..., -3, -2, \quad \omega \in \Omega . \tag{3.2}$$

Relations (3.1), (3.2) are suitable expressions of the expected extinction time $t_\ell(\omega)$, for every $\ell \in Z_+^* \setminus \{1\}$, $\ell \in Z_-^* \setminus \{-1\}$, of the corresponding circuit chain $(X_n)_{n\in N}$ describing uniquely a nonhomogeneous random walk with jumps (with steps – 1 or + 1or in the same position having a right elastic barrier at 0) in random environments through its unique representation by the directed circuits (c_k), (c'_k) and weights $(\mathrm{w}_k(\omega))$, $(\mathrm{w}'_k(\omega)), k \in Z_+$, $k \in Z_-$, respectively, for every random environment $\omega \in \Omega$.

REFERENCES

1. Ch. Ganatsiou, *On cycle representations of random walks in fixed, random environments,* Proceedings of the 58th World Congress of the International Statistical Institute, Dublin, 2011.
2. Ch. Ganatsiou, *On the study of transience and recurrence of the Markov chain defined by directed weighted circuits associated with a random walk in fixed environment,* Journal of Probability and Statistics, 2013 (2013), Article ID 424601, 5 pages.
3. Ch. Ganatsiou et al., *On the study of discrete-time birth-death circuit chains in random ergodic environments,* JP J. Biostatistics, **11**(2) (2014), 157-168.
4. Ch. Ganatsiou, *On the expected extinction time for the adjoint birth-death circuit chains in random environments,* 32nd European Meeting of Statisticians, Palermo, Italy, 22-26/7/2019.
5. Ch. Ganatsiou, *On the Study of Circuit Chains Associated with a Random Walk with Jumps in Fixed, Random Environments: Criteria of Recurrence and Transience*. In: Rassias T., Pardalos P. (eds) Mathematical Analysis and Applications. Springer Optimization and Its Applications, vol 154 (2019), Springer, pp. 185-203.
6. J. Grasman, *Stochastic epidemics: The expected duration of the endemic period in higher dimensional models*, Math. Biosci. 152(1998), pp.13-27.
7. S. Kalpazidou, Cycle Representations of Markov Processes, Springer, New York, NY, USA, 1995.
8. J. MacQueen, *Circuit processes,* Ann. Probab., **9**(1981), 604-610.
9. Qian Minping & Qian Min, *Circulation for recurrent Markov chains,* Z.Wahrsch. Verw. Gebiete, **59**(**2**)(1982), 203-210.
10. T. J. Newman, J. B. Ferdy and C. Quince, *Extinction times and moments closure in the stochastic logistic process*, Theoret. Population Biol., **65**(2004), 115-126.
11. M.A. Nowak, Evolutionary Dynamics: Exploring the Equations of Life, Harward University Press, 2006.
12. A. H. Zemanian, Infinite Electrical Networks, Cambridge University Press, Cambridge, UK, 1991.