Two-type Branching Random Walks with Different Configurations of Branching Sources

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Abstract

We consider branching random walks with two types of particles on multidimensional lattice. The underlying random walks for both types of particles are symmetric, homogeneous in space and irreducible. We assume that particles can die or produce offspring of both types at some lattice points, called branching sources. We consider the case when branching sources are located at every lattice point and the case when there is one branching source at the origin. We consider subpopulations of particles generated by a single particle of each type. We obtained the asymptotic behaviour for the first moments of subpopulations in both cases under different assumptions.

Keywords: branching random walks, two-type branching processes, multidimensional lattices

1. Introduction

The theory of branching random walks (BRWs) is widely studied for the last thirty years (see e.g. Gorostiza and Wakolbinger (1991), Albeverio et al. (1998)). BRWs can be used in describing processes with birth, death and transport of particles. However, results obtained in the theory of BRWs were mostly dedicated to the processes with one type of particle (see e.g. Getan et al. (2016), Molchanov and Whitmeyer (2017)). In contrast to the previous works in our paper we consider continuous-time symmetric multi-type branching processes on multidimensional lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$. The theory of multi-type branching processes was studied and described by different mathematicians (see e.g. Sevastyanov (1971)). In present work we mainly pay attention to BRWs with two types of particles with different configurations of generation centers, called branching sources, where particles can either die or produce offspring of both types.

2. BRW Model

We study continuos-time BRWs on $\mathbb{Z}^d$, $d \in \mathbb{N}$. The objects of the study are subpopulations of the particles which can be represented as the following column-vectors:

$$n_i(t,x,y) = [n_{i1}(t,x,y), n_{i2}(t,x,y)]^T, \quad i = 1,2.$$

Here $n_i(t,x,y)$ is the vector of particles at the point $y \in \mathbb{Z}^d$, generated by a single particle of type i which at time moment $t = 0$ was at the site $x \in \mathbb{Z}^d$. Its components $n_{ij}(t,x,y)$ are the numbers of particles at the point y of type j, generated by a single particle of type i at x at the moment $t = 0$. The initial condition is $n_{ij}(0,x,y) = \delta_i(j)\delta_x(y)$, where $\delta_u(v)$ is the Kronecker function on $\mathbb{Z}^d$ (or $\mathbb{R}$), that is if u, $v \in \mathbb{Z}^d$ (or $\mathbb{R}$)

$$\delta_u(v) = \begin{cases} 1, & u = v; \\ 0, & u \neq v. \end{cases}$$

Now consider what evolutions can happen with each particle. We assume that processes we study are Markov processes, so that each particle spends at every lattice point exponentionally distributed random time up to the first transformation. Then there are several evolutions. Firstly, each particle can die with the probability $\mu_i dt$, $i = 1,2$ during the small time period dt. Here $\mu_i \geqslant 0$ is the

mortality rate. Secondly, each particle can produce offspring of both types. We assume that the parental particle dies when it produces its offspring. Let $\beta_i(k,l)$, $i=1,2$, $k+l \geqslant 2$ be the intensity of particle of type i produce k offspring of the first type and l offspring of the second type. Thus, the corresponding probability to produce offspring during the time period dt is $\beta_i(k,l)dt$. Then we define the corresponding generating function of branching (without particle death) for $i=1,2$ (see Sevastyanov (1971)):

$$F_i(z_1,z_2) = \sum_{k+l\geqslant 2} z_1^k z_2^l \beta_i(k,l).$$

We assume that at every branching source particles of both types can produce their offspring. In what follows we consider two configurations of branching sources. In the first model we assume that branching sources are located at every lattice point. In the second model we consider the case when there is only one branching source. Without limitation of generality we can say that the branching source is located at the origin.

Finally, we assume that particles can jump between the lattice points. Let $\varkappa_i a_i(x,y)dt$, $i=1,2$ be the probability of the particle of type i jump from the point $x \in \mathbb{Z}^d$ to the point $y \in \mathbb{Z}^d$ during the time period dt. Here $\varkappa_i > 0$ is the diffusion coefficient. We also assume that the following conditions for intensities $a_i(x,y)$, $i=1,2$ hold for all x, $y \in \mathbb{Z}^d$: $a_i(x,y) = a_i(y,x)$ (*symmetry*), $a_i(x,y) = a_i(x-y,0)$ (*space homogeneity*). This two conditions allow us to replace the function $a_i(\cdot,\cdot)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ with the function $a_i(\cdot)$ on $\mathbb{Z}^d$ by the following formula $a_i(x,y) = a_i(y-x)$. Also we assume that underlying random walk for type $i=1,2$ is *irreducible*, so that $\operatorname{span}\{v \in \mathbb{Z}^d : a_i(v) > 0\} = \mathbb{Z}^d$. Besides, $\sum_{v\neq 0} a_i(v) = -a_i(0) = 1$. Then the generator of underlying random walk has the following form:

$$\mathcal{L}_i\psi(x) = \varkappa_i \sum_v \big[\psi(x+v) - \psi(x)\big] a_i(v).$$

In paper we are going to study the behaviour of random variables $n_{ij}(t,x,y)$, $i,j=1,2$, $x,y \in \mathbb{Z}^d$ in terms of their moments. We mainly pay attention to the bevaviour of the first moments of each subpopulation which we define as $m_{ij}(t,x,y) = \mathsf{E} n_{ij}(t,x,y)$. To obtain the results for the moments we introduce the following generating function of subpopulation $n_{ij}(t,x,y)$. Given $z=(z_1,z_2)$ we define

$$\Phi_i(t,x,y;z) = \mathsf{E} z_1^{n_{i1}(t,x,y)} z_2^{n_{i2}(t,x,y)}.$$

Later on we consider the generating function and the first moments in two cases with different configurations of branching sources.

3. BRW with Infinite Number of Branching Sources

Firstly, we consider the case when branching sources are located at every lattice point. Then in Makarova et al. (2020) there were obtained the differential equations of $\Phi_i(t,x,y;z)$ and $m_{ij}(t,x,y)$. As we are interested in studying the moments we shall consider the differential equation of $m_{ij}(t,x,y)$.

Lemma 0.1. *Let $\beta_i(k,l) \leqslant \frac{c_0^{k+l}}{k!l!}$, $i=1,2$, $k+l \geqslant 2$. Then, for each $i,j=1,2$, the functions $m_{ij}(t,x,y)$ satisfy the differential equation*

$$\begin{aligned}\frac{\partial m_{ij}(t,x,y)}{\partial t} &= \mathcal{L}_i m_{ij}(t,x,y) - \mu_i m_{ij}(t,x,y) - \sum_{k+l\geqslant 2} \beta_i(k,l) m_{ij}(t,x,y)\\ &\quad + \sum_{k+l\geqslant 2} \beta_i(k,l)(k m_{1j}(t,x,y) + l m_{2j}(t,x,y));\\ m_{ij}(0,x,y) &= \delta_i(j)\delta_x(y).\end{aligned}$$

In case when generators of underlying random walks are equal, so that $\mathcal{L}_1 = \mathcal{L}_2$, there were found explicit solutions of the equations for $m_{ij}(t,x,y)$. Here we want to consider the case when generators are different and have some additional properties.
Assume that the intensities $a_1(\cdot)$ and $a_2(\cdot)$ satisfy the following conditions:

$$\sum_v a_1(v)|v|^2 < \infty, \quad a_2(u) \sim \frac{H(u/|u|)}{|u|^{d+\alpha}}, \quad \alpha \in (0,2), \tag{1}$$

where $|\cdot|$ is the vector norm in $\mathbb{R}^d$ and $H(\cdot)$ is positive symmetric continuous function on $\{u \in \mathbb{R}^d : |u| = 1\}$. In this case we say that the underlying random walk for the particles of the first type has finite variance of jumps and for the particles of the second type — infinite variance of jumps.
Here in case of fixed space coordinates $x,\ y \in \mathbb{Z}^d$ we can obtain the asymptotic behaviour for $m_{ij}(t,x,y)$, $i,j = 1,2$ as t tends to infinity. We shal consider the results with regard to values

$$b := \sum_{k+l \geqslant 2} l\beta_1(k,l) \geqslant 0, \quad c = \sum_{k+l \geqslant 2} k\beta_2(k,l) \geqslant 0.$$

We study three cases

$$b = 0, c \geqslant 0 \quad \text{or} \quad b \geqslant 0, c = 0 \quad \text{or} \quad b > 0, c > 0.$$

Let

$$r_1 = \sum_{k+l \geqslant 2} (k-1)\beta_1(k,l) - \mu_1, \quad r_2 = \sum_{k+l \geqslant 2} (l-1)\beta_2(k,l) - \mu_2.$$

Then we obtain the following results with the usage of discrete Fourier transform which has the form

$$\widehat{f}(\theta) = \sum_{u \in \mathbb{Z}^d} e^{i(\theta,u)} f(u), \qquad \theta \in [-\pi,\pi]^d.$$

Case $b = 0, c \geqslant 0$**.** Here as $t \to \infty$

$$m_{11}(t,x,y) \sim e^{-\mu_1 t}\frac{\gamma_d}{t^{d/2}}; \qquad m_{21}(t,x,y) = 0;$$
$$m_{22}(t,x,y) \sim e^{-\mu_2 t}\frac{\gamma_{d,\alpha}}{t^{d/\alpha}}; \qquad m_{12}(t,x,y) = 0,$$

where γ_d and $\gamma_{d,\alpha}$ are positive constatns which depend on the lattice dimension.
Case $b = 0, c > 0$**.** Here as $t \to \infty$

$$m_{11}(t,x,y) \sim e^{-\mu_1 t}\frac{\gamma_d}{t^{d/2}}; \qquad m_{21}(t,x,y) \sim \begin{cases} ce^{-\mu_1 t}\frac{\gamma_d}{t^{d/2-1}}, & \text{if } -\mu_1 = r_2, \\ \frac{c}{-\mu_1 - r_2}\left(e^{-\mu_1 t} - e^{r_2 t}\right)\frac{\gamma_d}{t^{d/2}}, & \text{if } -\mu_1 \neq r_2; \end{cases};$$
$$m_{22}(t,x,y) \sim e^{r_2 t}\frac{\gamma_{d,\alpha}}{t^{d/\alpha}}; \qquad m_{12}(t,x,y) = 0.$$

Case $b > 0, c = 0$**.** Here as $t \to \infty$

$$m_{11}(t,x,y) \sim e^{r_1 t}\frac{\gamma_d}{t^{d/2}}; \qquad m_{21}(t,x,y) = 0;$$
$$m_{22}(t,x,y) \sim e^{-\mu_2 t}\frac{\gamma_{d,\alpha}}{t^{d/\alpha}}; \qquad m_{12}(t,x,y) \sim \begin{cases} be^{-\mu_2 t}\frac{\gamma_d}{t^{d/2-1}}, & \text{if } r_1 = -\mu_2, \\ \frac{b}{r_1 + \mu_2}\left(e^{r_1 t} - e^{-\mu_2 t}\right)\frac{\gamma_d}{t^{d/2}}, & \text{if } r_1 \neq -\mu_2; \end{cases}.$$

Case $b > 0, c > 0$**.** Here as $t \to \infty$

$$m_{11}(t,x,y) \sim \frac{e^{C_1 t}}{2C_2}\Big((r_1 - C_1 + C_2)e^{C_2 t} + (C_1 + C_2 - r_1)e^{-C_2 t}\Big)\frac{\gamma_d}{t^{d/2}};$$
$$m_{21}(t,x,y) \sim \frac{ce^{C_1 t}}{2C_2}\Big(e^{C_2 t} - e^{-C_2)t}\Big)\frac{\gamma_d}{t^{d/2}};$$
$$m_{22}(t,x,y) \sim \frac{e^{C_1 t}}{2C_2}\Big(C_1 + C_2 - r_1)e^{C_2 t} + (C_1 - C_2 - r_1)e^{-C_2 t}\Big)\frac{\gamma_d}{t^{d/2}};$$
$$m_{12}(t,x,y) \sim \frac{be^{C_1 t}}{2C_2}\Big(e^{C_2 t} - e^{-C_2 t}\Big)\frac{\gamma_d}{t^{d/2}},$$

where

$$C_1 = \frac{r_1 + r_2}{2}, \quad C_2 = \frac{((r_1 - r_2)^2 + 4bc)^{1/2}}{2}.$$

4. BRW with a Single Branching Source

In case when we have one branching source at the origin the differential equation for $m_{ij}(t,x,y)$ obtained in Lemma 3.1 is similar.

Lemma 0.2. *Let* $\beta_i(k,l) \leqslant \frac{c_0^{k+l}}{k!l!}$*, for all* $k+l \geqslant 2$*. Then, for each* $i,j = 1,2$*, the functions* $m_{ij}(t,x,y)$ *satisfy the differential equation*

$$\frac{\partial m_{ij}(t,x,y)}{\partial t} = \mathcal{L}_i m_{ij}(t,x,y) + \delta_0(x)\Big(-\mu_i m_{ij}(t,x,y) - \sum_{k+l\geqslant 2}\beta_i(k,l)m_{ij}(t,x,y)$$
$$+ \sum_{k+l\geqslant 2}\beta_i(k,l)(km_{1j}(t,x,y) + lm_{2j}(t,x,y))\Big);$$
$$m_{ij}(0,x,y) = \delta_i(j)\delta_x(y).$$

In this case we obtain the results with the usage of Laplace transform which has the form

$$Lf(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \quad \lambda \geqslant 0.$$

Let $p_i(t,x,y)$, $i = 1,2$ be the solution of the following Cauchy problems:

$$\frac{\partial p_i(t,x,y)}{\partial t} = \mathcal{L}_i p_i(t,x,y), \quad p_i(0,x,y) = \delta_x(y).$$

Define the Green functions of $p_i(t,x,y)$, $i = 1,2$ as follows

$$G_{\lambda,i}(x,y) := Lp_i(\lambda,x,y) = \int_0^\infty e^{-\lambda t} p_i(t,x,y)dt.$$

To simplify the formulae we assume that $\mu_1 = \mu_2 = 0$, $\beta_1(1,1) = \beta_2(1,1) =: \beta$. Let

$$\beta^2 G_{\lambda,1}(0,0)G_{\lambda,2}(0,0) \neq 1.$$

Then applying the Laplace and discrete Fourier transforms to the differential equations for $m_{ij}(t,x,y)$, $i,j=1,2$, we have

$$Lm_{ii}(\lambda,x,y)=\frac{\beta^2 G_{\lambda,i}(x,0)G_{\lambda,i}(0,y)G_{\lambda,k}(0,0)}{1-\beta^2 G_{\lambda,1}(0,0)G_{\lambda,2}(0,0)}+G_{\lambda,i}(x,y),\quad k=1,2,\quad k\neq i;$$

$$Lm_{ij}(\lambda,x,y)=\frac{\beta G_{\lambda,i}(x,0)G_{\lambda,j}(0,y)}{1-\beta^2 G_{\lambda,1}(0,0)G_{\lambda,2}(0,0)},\quad i\neq j$$

;

As in the previous Section we consider the case when the intensities $a_i(\cdot)$, $i=1,2$ satisfy (1). Here we found the representation of Laplace transform with regard to the lattice dimension d and parameter $\alpha\in(0,2)$ as $\lambda\to 0$. There were eight different cases. Here we present four cases.

Case 1. $d=1$, $\alpha\in(0,1/2]$. As $\lambda\to 0$

$$Lm_{11}(\lambda,x,y)\sim-\frac{1}{\beta^2 G_{\lambda,2}(0,0)};\quad Lm_{22}(\lambda,x,y)\sim-\frac{\sqrt{\lambda}}{\beta^2\gamma_1\sqrt{\pi}};$$

$$Lm_{21}(\lambda,x,y)\sim Lm_{12}(\lambda,x,y)\sim-\frac{1}{\beta}.$$

Case 2. $d=1$, $\alpha=1$. As $\lambda\to 0$

$$Lm_{11}(\lambda,x,y)\sim\frac{1}{\beta^2\gamma_{1,1}\ln\lambda};\quad Lm_{22}(\lambda,x,y)\sim-\frac{\gamma_{1,1}\ln\lambda}{\beta^2\gamma_1\sqrt{\pi}}$$

$$Lm_{21}(\lambda,x,y)\sim Lm_{12}(\lambda,x,y)\sim-\frac{\beta\gamma_1\sqrt{\pi}\gamma_{1,1}\ln\lambda}{\sqrt{\lambda}}.$$

Case 3. $d=2$, $\alpha\in[1,2)$. As $\lambda\to 0$

$$Lm_{11}(\lambda,x,y)\sim-\gamma_2\ln\lambda;\quad Lm_{22}(\lambda,x,y)\sim G_{\lambda,2}(0,0)$$

$$Lm_{21}(\lambda,x,y)\sim Lm_{12}(\lambda,x,y)\sim-G_{\lambda,2}(0,0)\beta\gamma_2\ln\lambda.$$

Case 4. $d=3$, $\alpha\in[3/2,1)$ or $d\geqslant 4$, $\alpha\in(0,2)$. As $\lambda\to 0$

$$Lm_{11}(\lambda,x,y)\sim\frac{G_{\lambda,1}(0,0)}{1-\beta^2 G_{\lambda,1}(0,0)G_{\lambda,2}(0,0)};\quad Lm_{22}(\lambda,x,y)\sim\frac{G_{\lambda,2}(0,0)}{1-\beta^2 G_{\lambda,1}(0,0)G_{\lambda,2}(0,0)};$$

$$Lm_{21}(\lambda,x,y)\sim Lm_{12}(\lambda,x,y)\sim\frac{\beta G_{\lambda,1}(0,0)G_{\lambda,2}(0,0)}{1-\beta^2 G_{\lambda,1}(0,0)G_{\lambda,2}(0,0)}.$$

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