Empirical analysis of branching random walks in random medium

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Abstract

This paper is devoted to the analysis of the time-continuous branching random walks in a random medium. Similar models are used in statistical physics, chemical kinetics, and population genetics. In the last twenty years, several limit theorems have been obtained for such processes. However, these theorems have some limitations, and the investigation of branching random walks on finite times seems impossible. This paper presents algorithms for the simulation of branching random walks in a random medium based on the Monte Carlo method. We obtained numerical estimations for models with various assumptions about the field structure, in particular for different branching potentials.

Keywords: Branching Random Walks, Random media, Monte-Carlo approach, Simulation of random processes

1. Introduction

Problems related to the intermittency in a random medium seem to have been first considered by Zeldovich et al in [1]. These problems were generalized and developed in two fundamental works by Gärtner and Molchanov in [2, 3]. In these works Gärtner and Molchanov strictly formalized the concept of intermittency and developed tools for studying the Anderson operator. These tools, in turn, make it possible to prove limit theorems for homogeneous branchind random walks (BRW) in a random medium, were done by Albeverio et al in [4]. This paper is currently the most complete and detailed analysis of BRW in a homogeneous random medium. An arising continuation of the results for the case of an non-homogeneous field was made in [5]. In addition we should mention a short course of lectures on random media by Molchanov [6], in which the main results of the theory of random media are presented in a concise form.

The structure of paper is as follows. In Section 2 we describe the model of BRW in both homogeneous and non-homogeneous case, as well as in both random and non-random medium. In Section 3 we present the previously obtasined auxiliary results and limit theorems. In Section 4 we explain the concept of intermittency and its empirical aproximations. Finally, in Section 5 we present the results of simulations of BRW with various initial conditions.

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2. Branching random walks in non-random and random medium

2.1. Non-random medium

Let us consider a evolving system of particles on a multidimensional lattice Z^d, $z \in N$. Particles can walk between points of the lattice and we assume that the probability of the jump from a point $x \in Z^d$ to a point $y \in Z^d$ during the small time h is equal to

$$a(x, y) \cdot h + o(h)$$

A random walk is assumed to be symmetric: $a(x, y) = a(y, x)$; homogeneous in space: $a(x, y) = a(0, y - x) =: a(y - x)$; regular: $\sum\limits_{z \neq 0} a(x) = -a(0)$; and irreducible: every point $x \in Z^d$ is reachable.

At first we assume homogeneity of branching medium. I.e., we assume the evolution of a particle at the point y during small time h consists of three opportunities. The particle can die with probability $b_0(y) \cdot h + o(h)$, where $b_0(y) \geqslant 0$. The particle can split into two particles with probability $b_2(y) \cdot h + o(h)$, where $b_2(y) \geqslant 0$. Nothing happens with the particle with probability $1+b_1(y) \cdot h + o(h)$, where $b_1(y) \leqslant 0$.

Let us assume for simplicity that at the initial moment of time there is exactly one particle on the lattice located at the point x. We combine walking and branching processes as follows. During small time h the evolution of a particle consists of several opportunities:

1. The particle can die with probability $b_0(y) \cdot h + o(h)$, where $b_0(y) \geqslant 0$;
2. The particle can split into two particles with probability $b_2(y) \cdot h + o(h)$, where $b_2(y) \geqslant 0$;
3. The particle can jump to the point $y + z$ with probability $a(z) \cdot h + o(h)$, where $a(z) \geqslant 0$;
4. Nothing happens with the particle with probability $1 + b_1(y) \cdot h + a(0) \cdot h + o(h)$, where $b_1(y) \leqslant 0$, $a(0) \leqslant 0$.

We will describe this system via the number of particles at time t at point y denoted by $\mu_t(y)$. We consider also the total population size $\mu_t := \sum_y \mu_t(y)$. This variables, in turn, will be studied in terms of their moments:

$$m_n(t, x, y) = \mathbb{E}_x \mu_t^n(y)$$

$$m_n(t, x) = \mathbb{E}_x \mu_t^n$$

We can abandon the assumption of the homogeneity of the medium. That is, assume that a branching process is possible only at a finite number of points. For simplicity, we assume that there is one such point and that it is located at the origin. For the models described above, a wide range of results was obtained, an overview of which can be found, for example, in [7].

2.2. Random medium

In a random medium the branching is governed by a pair of random variables $(b_0, b_2) = (b_0(y, \omega), b_2(y, \omega)), y \in \mathbb{Z}^d$. We assume that pairs $(b_0(y, \omega), b_2(y, \omega))$ are spatially i.i.d. Variable $\omega \in \Omega$, where $\Omega = \left(\mathbb{R}_+^2\right)^{\mathbb{Z}^d}$ represents sample realizations of field $(b_0(y, \omega), b_2(y, \omega))$. We consider branching takes place on a "frozen" medium, i.e. in a fixed medium realization ω. In a random medium the evolution of a particle at the point y during small time h consists of several opportunities:

1. The particle can die with probability $b_0(y, \omega) \cdot h + o(h)$, where $b_0(y, \omega) \geqslant 0$

2. The particle can split into two particles with probability $b_2(y, \omega) \cdot h + o(h)$, where $b_2(y, \omega) \geqslant 0$
3. The particle can jump to the point $y + z$ with probability $a(z) \cdot h + o(h)$, where $a(z) \geqslant 0$
4. Nothing happens with the particle with probability $1 + b_1(y, \omega) + a(0, \omega) \cdot h + o(h)$, where $b_1(y) \leqslant 0$,$a(0) \leqslant 0$.

We again assume that at the initial moment of time there is exactly one particle on the lattice located at the point x.

We introduce a random potential as

$$V(y, \omega) := b_2(y, \omega) - b_0(y, \omega)$$

Let us fix a medium realization ω. We denote by $\mu_t(y, \omega)$ the number of particles at time t at point y, and consider also the total population size $\mu_t := \sum_y \mu_t(y, \omega)$.

For every $n \in \mathbb{N}$ the "quenched" moments are introduced as:

$$m_n^p(t, x, y, \omega) = m_n^p(t, x, y) = [\mathbb{E}_x^\omega \mu_t^n(y)]^p\,;$$
$$m_n^p(t, x, \omega) = m_n^p(t, x) = [\mathbb{E}_x^\omega \mu_t^n]^p$$

We denote by $\langle\cdot\rangle$ the expectation with respect to the medium probability measure.
For every $n \in \mathbb{N}$ the "annealed" moments are determined as:

$$\langle m_n^p(t, x, y)\rangle;$$
$$\langle m_n^p(t, x)\rangle$$

We introduce the "quenched" generating functions as follows:

$$F(z; t, x, y, \omega) = F(z; t, x, y) := \mathbb{E}_x^\omega e^{-z\mu_t(y)}$$
$$F(z; t, x, \omega) = F(z; t, x) := \mathbb{E}_x^\omega e^{-z\mu_t}$$

Functions $F(z; t, x, y)$ and $F(z; t, x)$ satisfy the Skorohod equation for generating functions, see, e.g. [Albeverio et al, 2000]:

$$\partial_t F = AF + (b_2(x)F - b_0) \cdot (F - 1) \tag{1}$$

with the initial conditions:

$$F(z; 0, x, y) = e^{-z\delta_y(x);}$$
$$F(z; 0, x) = e^{-z}$$

From the equation (1) one can derive moments equations:

$$\partial_t m_n = Am_n + V(x)m_n + b_2(x)g_n\left[m_1, \ldots, m_{n-1}\right] \tag{2}$$

with the initial conditions:

$$m_n(0, \cdot, y) = \delta_y(\cdot); \qquad m_n(0, \cdot) \equiv 1,$$

where $g_1 \equiv 0$ and

$$g_n\left[m_1, \ldots, m_{n-1}\right] := \sum_{i=1}^{n-1} \binom{n}{i} m_i m_{n-i}, \text{ for } n \geqslant 2.$$

For the first moments, equations (2) take a simple form:

$$\partial_t m_1 = Am_1 + V(x)m_1$$

3. Feynman–Kac representation and limit theorems

Before proceeding to the description of limit theorems, we need a Feynman–Kac representation. For more information about its derivation, see [4]. Let us consider general equation for random fields:

$$\begin{aligned} \partial_t u(t,x) &= \varkappa \Delta u(t,x) + V(x,\omega)u(t,x), \\ u(0,x) &= u_0(x) \end{aligned} \tag{3}$$

Let $V(x,\omega)$ be i.i.d. such that $\left\langle \left(\frac{V^+(0)}{\ln_+ V^+(0)}\right)^d \right\rangle < \infty$, where $\ln_+(x) := \max(\ln(x), 1)$. Let $\limsup\limits_{|x|\to\infty} \frac{\ln_+ u_0(x)}{|x| \ln |x|} < 1$. Under these conditions equation (3) has unique non-negative solution, which admits the Feynman–Kac representation:

$$u(t,x) = \mathbb{E}_x \left[\mathrm{e}^{\int\limits_0^t V(x_s,\omega)ds} u_0(x_t) \right],$$

where x_t is underlying random walk.

Recall that

$$\partial_t m_1 = A m_1 + V(x) m_1$$

with the initial conditions

$$m_1(0,\cdot,y) = \delta_y(\cdot); \qquad m_1(0,\cdot) \equiv 1$$

Then we can apply the Feynman-Kac representation and obtain:

$$m_1(t,x,y) = \mathbb{E}_x \left[\mathrm{e}^{\int\limits_0^t V(x_s,\omega)ds} \delta_y(x_t) \right]$$

$$m_1(t,x) = \mathbb{E}_x \left[\mathrm{e}^{\int\limits_0^t V(x_s,\omega)ds} \cdot 1 \right]$$

Assume that

$$\lim_{t\to\infty} \frac{t}{\ln \langle \mathrm{e}^{pVt} \rangle} = 0.$$

Then for the annealed moments $\langle m_1(t,x,y) \rangle$ and $\langle m_1(t,x) \rangle$ it can be shown that:

$$\lim_{t\to\infty} \frac{\ln \langle m_1^p \rangle}{\ln \langle \mathrm{e}^{pVt} \rangle} = 1$$

$$\lim_{t\to\infty} \frac{\ln \langle m_n^p \rangle}{\ln \langle \mathrm{e}^{pnVt} \rangle} = 1$$

4. The intermittency

Given two functions $f, g : \mathbb{R}_+ \to \mathbb{R}$, we will write $f \ll g$ if $g(t) - f(t) \to \infty$ as $t \to \infty$.
Let $\{\eta(t, x); x \in \mathbb{Z}^d\}$ be a family of non-negative spatially homogeneous random fields and

$$\Lambda_p(t) = \ln \langle \eta(t, 0)^p \rangle < \infty, \quad t \geq 0, p \in \mathbb{N}$$

The random fields $\{\eta(t, x); x \in \mathbb{Z}^d\}$ are called intermittent Gärtner and Molchanov [2] if they are ergodic and

$$\Lambda_1(t) \ll \frac{\Lambda_2(t)}{2} \ll \frac{\Lambda_3(t)}{3} \ll \dots \tag{4}$$

Note that if so-called Lyapunov exponents $\lambda_p = \lim\limits_{t\to\infty} \frac{\Lambda_p(t)}{t^\beta}, p \in \mathbb{N}$ exist for some $\beta \geqslant 1$, and $\lambda_1 < \frac{\lambda_2}{2} < \frac{\lambda_3}{3} < \dots$, then condition (2) will be met.

Intermittency means that as $t \to \infty$ the main contribution to each moment function is carried by higher and higher and more and more widely spaced "peaks" of the random field.

Let for simplicity consider $\langle m_1^p \rangle$, and assume that potential has the Weibull-type tails:

$$\lim_{z\to+\infty} (1 - F_V(z)) = \mathrm{e}^{-cz^{-\alpha}},$$

where $\alpha > 1$, $c > 0$.

In this case based on the previous theorem one can derive the following:

$$\lim_{t\to\infty} \frac{\ln \langle m_1^p \rangle}{t^{\frac{\alpha}{\alpha-1}}} = C \cdot p^{\frac{\alpha}{\alpha-1}}$$

Then we can observe that

$$\frac{\lambda_p^{m_1}}{p} = \lim_{t\to\infty} \frac{\ln \langle m_1^p \rangle}{t^\beta} = C \cdot p^{\frac{1}{1-\alpha}}$$

Therefore, the sequence of $\frac{\lambda_p^{m_1}}{p}$ is strictly increasing as function of p, which, in turn, implies the intermittency of $\langle m_1^p \rangle$. The same result is valid for $\langle m_n^p \rangle$, where $n \in \mathbb{N}$.

5. Simulation

For the simulation, we used the R 3.5.1 data analysis environment. Parallel programming with 47 cores was used. A simple one-dimensional random walk was considered. The intensity a_0 for all models is assumed to be -1. That is, the particle waits for an exponential time, and then moves equally likely to one of the neighboring points. The simulated BRW have the potential to experience an exponential explosion [7]. Therefore, the simulation of the process does not take place over a finite period of time, but until there are 1000 particles on the lattice at the same time. In addition, an exponential approximation was used: as long as at least 90% of the models has at least one particle but less than 1000 particles, the remaining exploded 10% are imputed using a generalized linear regression of the logarithm of the number of particles on time. The coefficient of determination of such imputation was 90%.

Здесь пока считается. Будет три варианта моделей в неоднородных средах: неслучайная среда, случайная с вейбулловским хвостом, случайная с тройным экспоненциальным хвостом. Для однородной среды просто скажу, что все то же самое.

Что не сделано: можно ли придумать статистический способ оценки перемежаемости? Чтобы не глазами на графики смотреть.

6. Conclusion

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