



Risk and Return of the Tontine: A Brief Discussion

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Abstract:

This article analyzes the stochastic aspects of a tontine using a Gompertz distribution. In particular, the probabilistic and demographic risks of a tontine investment are examined. The expected value and variance of tontine payouts are calculated. Both parameters increase with age. The stochastic present value of a tontine payout is compared with the present value of a fixed annuity. It is shown that only at very high ages the tontine is more profitable than an annuity. Finally, the demographic risks associated with a tontine are discussed. Elasticities are used to calculate the impact of changes in modal age on the tontine payout. It is shown that the tontine payout is very sensitive to changes in modal age.

Keywords: Gompertz distribution, life table, annuity, demography, mortality.

1. Gompertz Distribution

Benjamin Gompertz proposed in 1825 a life table function, which is one of the oldest and most famous models of demography. It states that the mortality intensity exponentially increases with age in adulthood. It has been much applied in the life table analysis and insurance mathematics with various modifications (e.g., Gompertz-Makeham law). Due to declining children and youth mortality, it has again become essential in order to describe "modern" life tables with low mortality (cf. Pollard, 1998 or Pflaumer, 2018). The model allows to fully describe the present and future life tables in industrialized countries by using only two parameters, in principle, which are both easy to estimate from data. The Gompertz model provides a good approximation of life tables in these populations (see, e.g., Pflaumer, 2018).

The survival function is given by

$$l(x) = \exp\left(e^{-k \cdot m} - e^{k \cdot (x-m)}\right), \text{ where } m \text{ is the modal age and } k \text{ the growth rate of the force of mortality}$$

function $\mu(x) = k e^{k(x-m)}$. The life expectancy at birth is calculated by $e_0 = \frac{\gamma - m}{k}$ with $\gamma = 0,577221566\dots$ (Euler-Mascheroni-Constant).

The rectangularization of the survival curve is defined as a trend toward a more rectangular shape of the survival curve, due to increased survival and concentration of deaths around the mean age at death. The variability in the age at death declines and deaths are being compressed into the upper years of life. An increase in the parameter k causes rising rectangularization.

2. Tontine

A tontine is a rising life annuity, in which the annual constant interest yield or dividend of the total investment sum is paid out to the surviving subscribers of the tontine. It is commonly believed that the Italian banker Lorenzo Tonti invented tontines in the 17th century. In the 18th and 19th centuries, tontines were a popular means of public financing (see, e.g., Hellwege, 2018). The tontine ends with the death of the last subscriber. The capital falls to the initiator of the tontine. Recently, the tontine is considered as an attractive alternative to life annuities (see, e.g., Milevsky, 2014 or Hellwege, 2018). All calculations and graphs were performed with R. (<https://www.R-project.org/>).

Formulas of the tontine payout

It is assumed that all 400 investors buy at age $u=65$. The tontine share per investor is 100. The mortality follows a Gompertz distribution with $m=88.7$ and $k=0.1152$ (data for estimation: German life table for females (2013/2015)).

- l(x): Probability of surviving from birth to age x
- l(u): Probability of surviving from birth to age u (l(65)= 0.93691)
- N(x,u): Number of investors at age x alive who buy at age u (u=65)

$$N(0,u) = \frac{n}{l(u)} \text{ Number at age } x=0 \text{ of the life table (427)}$$

- Z: Total yearly payout (1,600)
- n=N(u,u): Number of subscribers of the tontine at age u (400)

Z / n : Initial payout (4)

t(x,u): (Expected) tontine payout per surviving investor at age x $x \geq u$

$$t(x,u) = \frac{Z}{N(x,u)} = \frac{Z}{n \cdot \frac{l(x)}{l(u)}} = \frac{Z}{\frac{n}{l(u)} \cdot \exp(e^{-k \cdot m} - e^{k \cdot (x-m)})} = \frac{Z}{n \cdot \exp(e^{k \cdot (u-m)} - e^{k \cdot (x-m)})}; \quad x \geq u$$

The growth rate of the tontine is $\frac{\delta t}{t} = \mu(x) = k \cdot e^{k \cdot (x-m)}$

t(x,u) is a random variable, since it is assumed that N(x,u) has a binomial distribution (see Li; Tuljapurkar, 2013). The variance is:

$$\begin{aligned} \text{Var}(N(x,u)) &= n \cdot \exp(e^{k \cdot (u-m)} - e^{k \cdot (x-m)}) \cdot (1 - \exp(e^{k \cdot (u-m)} - e^{k \cdot (x-m)})) \\ &= n \cdot (\exp(e^{k \cdot (u-m)} - e^{k \cdot (x-m)}) - \exp(2 \cdot e^{k \cdot (u-m)} - 2 \cdot e^{k \cdot (x-m)})) \end{aligned}$$

The 0.05- and 0.95-quantiles of N(x,u) are $N^*(x,u)_{0,05}$ and $N^*(x,u)_{0,95}$, which are obtained from the corresponding binomial distribution.

The lower and upper limits of a 90%-confidence interval of the tontine payouts are given by the quantiles

$$L_{x,0.05} = \frac{Z}{N^*(x,u)_{0,95}} \quad \text{and} \quad U_{x,0.05} = \frac{Z}{N^*(x,u)_{0,05}}.$$

Approximative $(1 - \alpha)$ - confidence interval for the survivors:

$$N(x,u) \pm u_{\frac{1-\alpha}{2}} \sqrt{\text{Var}(N(x,u))}$$

Approximative $(1 - \alpha)$ - confidence interval for the tontine payouts.

$$\text{Lower limit: } \frac{Z}{N(x,u) + u_{\frac{1-\alpha}{2}} \sqrt{\text{Var}(N(x,u))}}; \quad \text{Upper Limit: } \frac{Z}{N(x,u) - u_{\frac{1-\alpha}{2}} \sqrt{\text{Var}(N(x,u))}}$$

u_α is the α -quantile of the standard normal distribution.

Present value of the tontine payout up to age a (with a first payout at age u+1):

$$\int_u^a e^{-i(u-x)} \cdot \frac{Z}{n \cdot \exp(e^{k \cdot (u-m)} - e^{k \cdot (x-m)})} dx = \frac{Z}{n} \cdot \exp(i \cdot u - e^{k \cdot (u-m)}) \cdot \int_u^a \exp(e^{k \cdot (x-m)} - i \cdot x) dx$$

Present value of the tontine payout up to age a (with a first payout at age u):

$$\int_u^a e^{-i(u-x)} \cdot \frac{Z}{n \cdot \exp(e^{k \cdot (u-m)} - e^{k \cdot (x-m)})} dx + \frac{Z}{n} = \frac{Z}{n} \cdot \left(1 + \exp(i \cdot u - e^{k \cdot (u-m)}) \cdot \int_u^a \exp(e^{k \cdot (x-m)} - i \cdot x) dx \right)$$

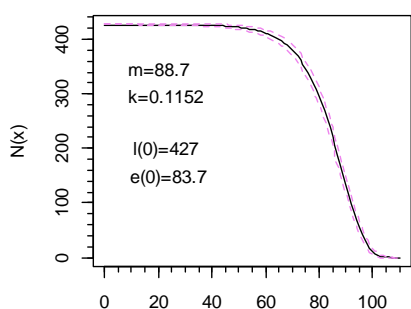


Fig. 1: Survivor Funktion from Age 0 with 90%-Confidence Interval

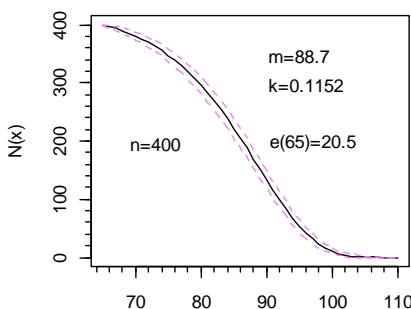


Fig. 2: Survivor Funktion from Age 65 with 90%-Confidence Interval

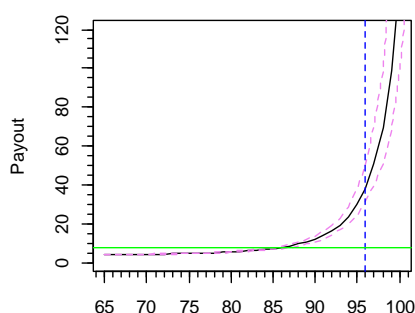


Fig. 3: Annuity (green) and Tontine Payout with 90%-Confidence Interval

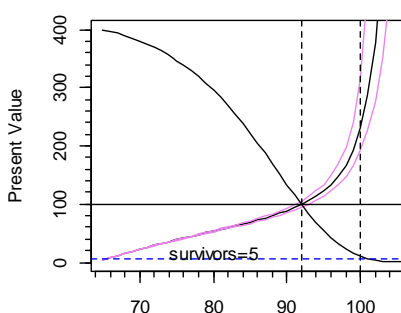


Fig. 4: Present Value of the Tontine with 90%-Confidence Interval

Figures 1 and 2 show the survivor functions with 90%-confidence intervals. In Figure 3 the initial payment (or contribution) to the tontine is 4 and the comparable annuity is 7.3 ($\ddot{a}_y(65)=13.734$ at 4%). At age 86 the tontine payout exceeds the annuity of 7.3. At age $x=96$, e.g., the payout varies between 30.8 and 50.0; expected 38.1. In Figure 4 the present value of tontine payouts exceeds at age 92 the initial investment of 100. The present value increases rapidly after that age with increasing confidence intervals.

3. Demographic Risks

The demographic risk is explained by a rise in life expectancy or in modal age.

Elasticity of $t(x,u)$ with respect to m :

$$\varepsilon_{t,m}(x) = \frac{\delta t}{t} \cdot m = \left(-k \cdot e^{k(x-m)} + k \cdot e^{k(u-m)}\right) \cdot m = \left(-r(x) + k \cdot e^{k(u-m)}\right) \cdot m$$

$$m \gg u : \rightarrow \hat{\varepsilon}_{t,m}(x) = -r(x) \cdot m ; \quad \varepsilon_{t,m}(m) \approx -k \cdot m$$

Elasticity of $t(x,u)$ with respect to k

$$\varepsilon_{t,k}(x) = \frac{\delta t}{t} \cdot k = k \cdot e^{k(x-m)} \cdot (x-m) + k \cdot e^{k(u-m)} \cdot (m-u) = r(x) \cdot (x-m) + k \cdot e^{k(u-m)} \cdot (m-u)$$

Approximation: $k \cdot e^{k(x-m)} \cdot (x-m) = r(x) \cdot (x-m)$

$$\varepsilon_{t,k}(m) = k \cdot e^{k(u-m)} \cdot (m-u) > 0 ; \quad \varepsilon_{t,k}(x) \text{ has a minimum at } x = m - \frac{1}{k}$$

Approximate solution: $\varepsilon_{t_s,k}(x)$ has a zero point at $m - e^{k(u-m)} \cdot (m-u)$ (Taylor series of order 1 around $x=m$). From the total derivative ($x>m$) $dt = f_m \cdot dm + f_k \cdot dk = 0$ follows

$$\frac{dk}{dm} = -\frac{f_m}{f_k} = \frac{k \cdot (e^{k \cdot x} - e^{k \cdot u})}{e^{k \cdot x} \cdot (x-m) + e^{k \cdot u} \cdot (m-u)} \text{ or } \Delta k \approx \frac{k \cdot (e^{k \cdot x} - e^{k \cdot u})}{e^{k \cdot x} \cdot (x-m) + e^{k \cdot u} \cdot (m-u)} \cdot \Delta m.$$

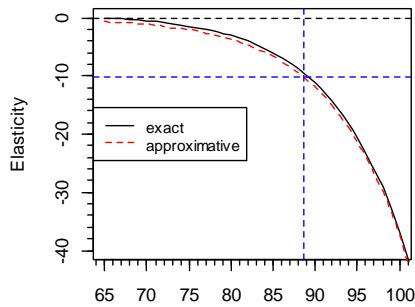


Fig. 5: Tontine Elasticity with respect to m

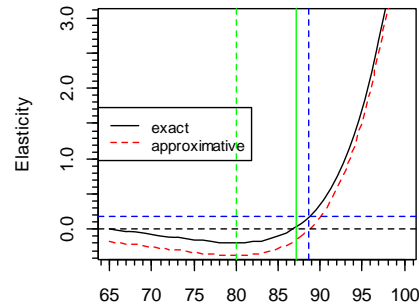


Fig. 6: Tontine Elasticity with respect to k

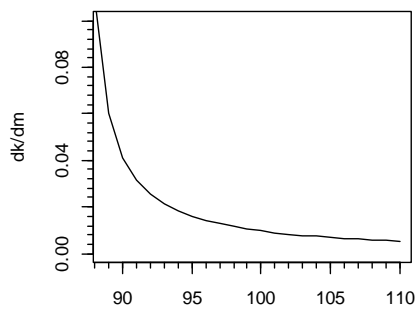


Fig. 7: Derivative of k with respect to m

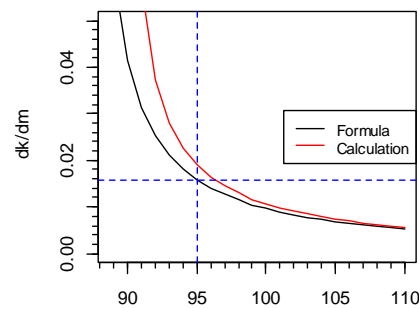


Fig. 8: Derivative and Numerical Calculation with $\Delta m=1$

At the modal age of 88.7, the elasticity is -10 (see Fig. 5); an increase of the modal age reduces the tontine payout substantially (e.g., an increase of m by 1% reduces the tontine payout by about 10% at age 88.7). The reduction can be partly compensated by an increase of k , which means a rectangularization of the life table (see Fig. 6 and 7). Figure 8 demonstrates the impact of m on k . For example, if at age $x=95$ the modal age increases by 1 year, k would have to increase by nearly 0.02 for the payout to remain the same. This increase in k is unrealistically high. Therefore, an increase in m will always lead to a reduction in the payout.

4. Conclusion

A tontine is a worthwhile investment for people who reach a very old age. But it is subject to three types of risk: financial (interest rate, default, inflation), probabilistic and demographic. The probabilistic risk can be reduced by increasing the number of investors. The demographic risk should not be underestimated. It can lead to a significant reduction in tontine payout.

Literature

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