Sequential criticality test for branching process with immigration

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Abstract

We consider a sequential criticality test (SCT) for branching process with immigration. Observations are collected sequentially as time goes by. Using a stopping time based on the observed Fisher information, SCT is found to be a Z-test for local alternatives including sub- and super-critical hypotheses. The joint density and Laplace transform of the test statistics and stopping time are obtained from the joint Laplace transform with respect to a Bessel process driven by Dambis-Dubins-Schwartz (DDS) Brownian motion. The joint density can be expressed as a formula including parabolic cylinder functions. Since the joint density are not suitable for computing the joint moments, we obtain the joint Laplace transform. Numerical studies are conducted to verify our asymptotic results.

Keywords and phrases

Observed Fisher information; DDS Brownian motion; Bessel process; Joint Laplace transform; Z-test

1 Introduction and model setting

We consider sequential test for the criticality of branching processes with immigration. Suppose $\{\xi_{n,k}\}_{n,k\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are independent, nonnegative, integer-valued random variables with mean and variance (m,σ^2) and (λ,σ_Y^2) respectively. Let $\{Z_n\}$ be the nth generation size of a Galton-Watson processes;

$$Z_n=\sum_{k=1}^{Z_{n-1}}\xi_{n,k}+Y_n,\quad n\in\mathbb{N},\tag{1}$$

where $\xi_{n,k}$ is the number of offspring of the kth individual belonging to nth generation and Y_n is the number of the immigration in the nth generation. The initial value Z_0 is a random variable which is independent of $\{\xi_{n,k}\}$. Sriram, Basawa, and Huggins (1991) considered the fixed accuracy sequential estimation of offspring mean m using the stopping time based on the observed Fisher information. We consider a sequential testing problem for criticality of m.

We consider models with local parameters;

$$\text{Under}\quad P^0:(m,\sigma^2)=(1,\sigma^2),\quad \text{Under}\quad P^\delta:(m,\sigma^2)=(1+\delta/\sqrt{c},\sigma_c^2),$$

with assumption $\sigma_c^2\to\sigma^2$, as $c\to\infty$. The hypotheses for sub- and super-critical tests are

$$H_0:\delta\geq 0\quad \text{vs}\quad H_1:\delta<0,\qquad H_0:\delta\leq 0\quad \text{vs}\quad H_1:\delta>0.\tag{2}$$

Suppose we have a sample (Z_n,Y_n), $n=1,2,\dots,N$ from (1). When $\left\{\xi_k^{(n)}\right\}$ and $\{Y_n\}$ have power series distributions $P\left(\xi_k^{(n)}=j\right)=a_j\theta^j/A(\theta)$ and $P(Y_n=k)=b_k\phi^k/B(\phi)$, the M.L.E. $\hat{m}_N$ and the observed Fisher information of m are

$$\hat{m}_N = \sum_{n=1}^{N} (Z_n - Y_n) / \sum_{n=1}^{N} Z_{n-1}, \quad I_N(m) = \sum_{n=1}^{N} Z_{n-1}/\sigma^2. \tag{3}$$

Define a stopping time based on the observed Fisher information of m; for $c > 0$,

$$\tau_c \equiv \inf\left\{ N > 1 : \sum_{n=1}^{N} Z_{n-1}/s_N^2 \geq c \right\}. \tag{4}$$

where s_N^2 is a natural estimator of σ^2 written as

$$s_N^2 = \sum_{n=1}^{N} 1_{\{Z_{n-1}>0\}} (Z_n - Y_n - \hat{m}_N Z_{n-1})^2 / (N Z_{n-1}). \tag{5}$$

In (4), c controls for the accuracy of estimation, which could be predetermined by empirical researchers. The test statistics is defined as

$$\hat{\delta}_{\tau_c} \equiv \sqrt{c}\,(\hat{m}_{\tau_c} - 1). \tag{6}$$

We investigate the asymptotic properties of the sequential testing procedure $(\hat{\delta}_{\tau_c}, \tau_c)$.

2 Continuous-time model and joint asymptotic behavior of $(\hat{\delta}_{\tau_c}, \tau_c)$

Next we approximate the above discrete-time models to the continuous-time models. Let $m_c = 1 + \delta/\sqrt{c} \to 1$ and $\sigma_c^2 \to \sigma^2$ as $c \to \infty$. We also let $Z_0/\sqrt{c} \to x_0$ as we would like to know the effect of initial value. Then, the branching process (1) converges to a Feller process

$$Z_{\lfloor \sqrt{c}t \rfloor}/\sqrt{c} \Rightarrow X_t \equiv x_0 + \sigma \int_0^t \sqrt{X_s} dW_s + \delta \int_0^t X_s ds + \lambda t \tag{7}$$

where "$\Rightarrow$" stands for weak convergence and W is a standard Brownian motion. For $\delta = 0$, letting $q_t = 4X_t/\sigma^2$, we obtain the squared Bessel process with dimension $4\lambda/\sigma^2$;

$$q_t = q_0 + 2\int_0^t \sqrt{q_s} dW_s + 4\lambda t/\sigma^2. \tag{8}$$

The hypotheses (2) are reduced to in continuous time;

$$\begin{aligned} &\text{Under} \quad P^0 : dX_t = \sigma\sqrt{X_t} dW_t + \lambda dt, \\ &\text{Under} \quad P^\delta : dX_t = \sigma\sqrt{X_t} dW_t + (\delta X_t + \lambda)\, dt. \end{aligned}$$

Using a Girsanov transformation $d\tilde{W}_t = -\delta X_t dt/\sigma + dW_t$, the likelihood process is represented as

$$dP^\delta/dP^0 = \exp\left(\delta \int_0^t \left(\sqrt{X_s}/\sigma\right) dW_s - \delta^2/2 \int_0^t X_s/\sigma^2 ds \right).$$

Then, we have the M.L.E. and observed Fisher information of δ,

$$\tilde{\delta}_t = \delta + \sigma \int_0^t \sqrt{X_s} d\tilde{W}_s / \int_0^t X_s ds, \quad \tilde{I}_t = \int_0^t X_s/\sigma^2 ds,$$

which corresponded to the limit of the $N(\hat{m}_N - 1)$ and the observed Fisher information in discrete time in (3) with $t = 1$.

Define martingale M_t and its quadratic variation $\langle M\rangle_t$ as

$$M_t \equiv \int_0^t \sqrt{X_s/\sigma^2} dW_s, \quad \langle M\rangle_t = \int_0^t X_s/\sigma^2 ds. \tag{9}$$

According to Theorem 7.2 in Ikeda and Watanabe (1989) p.85, letting

$$U_v \equiv \inf\left\{t \geq 0 : \langle M\rangle_t = v\right\} = \langle M\rangle_v^{-1}, \tag{10}$$

$\langle M\rangle_{U_v} = v$ and $B_v \equiv M_{U_v}$ becomes a Brownian motion. B_v is so-called a time-changed (or DDS) Brownian motion. Let

$$\rho_v \equiv X_{U_v}/\sigma^2 = d\langle M\rangle/dt|_{t=U_v}, \tag{11}$$

then we can obtain the main theorem as follows.

Theorem 1. *Suppose Z_n is generated by the model (1) with an initial value Z_0 satisfying $Z_0/\sqrt{c} \to x_0$. Then the asymptotic behavior of the stopping times τ_c in (4) and the sequential test statistics $\hat{\delta}_{\tau_c}$ is given as follows: as $c \uparrow \infty$,*

$$\left(\hat{\delta}_{\tau_c}, \tau_c/\sqrt{c}\right) \Rightarrow \left(\delta + \int_0^{U_1} X_s dW_s, U_1\right) = \left(\delta + B_1, \int_0^1 \rho_s^{-1} ds\right) \tag{12}$$

where B_t is a standard Brownian motion, $U_1 \equiv \inf\left\{t : \int_0^t X_s/\sigma^2 ds = 1\right\}$, and ρ_t is the Bessel process with drift δ, dimension $d = 2\lambda/\sigma^2 + 1$, and initial value $\rho_0 = x_0/\sigma^2$;

$$d\rho_t = \left(\frac{\lambda/\sigma^2}{\rho_t} - \delta\right)dt + dB_t. \tag{13}$$

The joint Laplace transform of (ρ_v, U_v) under H_0 can be obtained from the time change of the squared Bessel process q_t in (8) with $q_0 = 4x_0/\sigma^2 = 4\rho_0$;

$$\int_0^\infty e^{-\gamma v} E_{q_0}^0 \left[\exp\left(-\alpha\rho_v - \beta U_v\right)/\rho_v\right] dv = \int_0^\infty e^{-\beta t} E_{q_0}^0 \left[\exp\left(-\frac{\alpha}{4} q_t - \frac{\gamma}{4}\int_0^t q_s ds\right)\right] dt. \tag{14}$$

Using the Bessel bridge in Pitman and Yor (1982), under H_0 we can obtain,

$$\mathrm{P}_{q_0}\left(\int_0^u \rho_s^{-1} ds \in dt, \rho_u \in dz\right) = \frac{z^{\nu+1}}{q_0^\nu} \mathrm{is}_u\left(2\nu, t/2, 0, q_0 + z, \sqrt{q_0 z}\right) dt dz,$$

with $\nu = d/2 - 1 = \lambda/\sigma^2 - 1/2$. is_u function is special inverse Laplace transform defined as

$$\mathrm{is}_u(\nu, t, r, z, x) = \mathscr{L}_\gamma^{-1}\left[\left(\frac{\sqrt{2\gamma}}{\sinh\left(t\sqrt{2\gamma}\right)}\right) \exp\left(-r\sqrt{2\gamma} - z\sqrt{2\gamma}\coth\left(t\sqrt{2\gamma}\right)\right) I_\nu\left(\frac{2x\sqrt{2\gamma}}{\sinh(t\sqrt{2\gamma})}\right)\right].$$

where I_ν is the modified Bessel function. See Borodin and Salminen (2002) for the expression of is_u function which includes parabolic cylinder functions. Using Girsanov's theorem, we can obtain the joint probability densities of (ρ_v, U_v) under H_1 with initial value.

The joint Laplace transform can also be obtained as

$$E_{q_0}^0\left[\exp\left(-\alpha\rho_v - \beta U_v\right)/\rho_v\right] = \sum_{n=0}^\infty \sum_{j=0}^\infty \sum_{l=0}^\infty \frac{q_0^n \alpha^j \beta^l}{n! j! l!} \int_0^1 J(t, v, n, j, l) dt, \tag{15}$$

with

$$J(t, v, n, j, l) = \frac{(-1)^n \left(1 - \sqrt{s}\right)^{j+n} \left(\sqrt{s} + 1\right)^{-j-\nu-n-1} s^{\frac{\nu-3}{4}} \log^l(s)(-n-\nu-1)^{(j)}}{\Gamma\left(\frac{1}{2}(j + l - n + 1)\right)}$$
$$\times 2^{\frac{1}{2}(-j-3l-3n-1)+\nu} v^{\frac{1}{2}(j+l-n+1)-1} {}_2F_1\left(-j, -n; -j-n-\nu; \frac{\left(\sqrt{s}+1\right)^2}{\left(\sqrt{s}-1\right)^2}\right)$$

where $x^{(m)}$ is the factorial power and ${}_2F_1$ is Gauss hypergeometric function. Using Girsanov's theorem, we can obtain the joint Laplace transform of (ρ_v, U_v) under H_1.

3 Simulation results

In Monte Carlo simulation, we let $\xi_n^{(k)} \sim i.i.d.$ Negative Binomial (k, p) and $Y_n \sim i.i.d.$ Poisson (λ) replication= 10,000 and set initial value $x_0 = 0, 1$, $m = 0.99, 1, 1.01$, $k = 5, \lambda = 10$. It is easy to show $p = k/(m+k)$ and $\sigma^2 = m + m^2/k$. The table provides rejection rates (RRs), and the the operating characteristics (OCs): means and standard deviations of τ_c and $\hat{m}_{\tau_c}$ with theoretical values in parentheses. The rejection rates are close to the theoretical values obtained from the standard normal table. Although the mean stopping times are foretasted well by the numerical values from the joint Laplace transform, their theoretical standard deviations turn out to be smaller than the simulation results. We also note that $\hat{m}_{\tau_c}$ can be estimated with the standard deviations equal to the fixed accuracy $1/\sqrt{c}$.

Table 1: Rejection rates (RRs) and OCs: $m = 0.99, 1, 1.01$

		$x_0 = 0$						$x_0 = 1$					
m	$\sqrt{c}$	RR(%)		τ_c		$\hat{m}_{\tau_c}$		RR(%)		τ_c		$\hat{m}_{\tau_c}$	
		Left	Right	Mean	sd	Mean	sd	Left	Right	Mean	sd	Mean	sd
1	100	5.9	5.6	49.0	7.0	1.00	0.011	5.7	5.6	40.0	7.5	1.00	0.011
				(49.7)	(5.0)		(0.01)			(40.7)	(4.7)		(0.01)
	200	5.1	5.5	98.9	12.2	1.00	0.0051	5.5	5.5	80.6	12.1	1.00	0.0051
				(99.5)	(10.0)		(0.005)			(81.4)	(9.3)		(0.005)
0.99	100	26.4		52.7	8.15	0.99	0.0105	26.7		43.5	8.40	0.99	0.0104
		(26.0)		(54.2)	(5.72)		(0.01)	(26.0)		(44.9)	(5.41)		(0.01)
	200	64.0		116.9	15.64	0.99	0.0051	63.8		97.8	15.91	0.99	0.0051
		(63.9)		(118.8)	(13.04)		(0.005)	(63.9)		(99.8)	(12.53)		(0.005)
1.01	100	26.7		45.7	6.41	1.01	0.0106	26.8		36.9	6.63	1.01	0.0106
		(26.0)		(45.9)	(4.43)		(0.01)	(26.0)		(37.2)	(4.05)		(0.01)
	200	63.7		85.7	9.72	1.01	0.0051	62.4		68.4	9.49	1.01	0.0051
		(63.9)		(85.4)	(7.85)		(0.005)	(63.9)		(68.4)	(7.05)		(0.005)

4 Conclusion

We develop the sequential testing method of near criticality hypothesis for branching process with immigration. We consider diffusion approximations and derive the asymptotic results for OC's by using the time change methods under the null. The joint density and Laplace of the stopping time and the sequenital test statisitics under the null can be transformed to the joint density and Laplace transform under the local alternatives via Girsanov's theorem. OC's can be computed by using the joint densities and Laplace transform of Bessel processes.

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