Corner pillars of probability literacy

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Abstract:

Approaches towards teaching probability have focussed on the frequentist interpretation of probability. For reasons of simplicity, this meaning has also been the basis of more recent suggestions of informal inference to enable learning paths to inference. Yet, there are investigations such as Carranza and Kuzniak (2009) that provide empirical evidence about the shortcomings in understanding probability and its scope if the interpretation of probability neglects the full spectrum of historically grown aspects. Inference is easier to understand from integrating Bayesian aspects and risk. There are several corner stones that may provide learning paths that lead to more sustainable probabilistic intuitions. We identify them as 'Beginning early and developing the ideas and concepts spirally', 'Use games intelligently to let sustainable probabilistic intuitions grow', 'Build thinking in Bayesian and risk terms as early as possible', 'Connect probability with statistical inference right from the beginning', and 'Develop the twin relation between probability and risk to clarify the purpose of probability'. The first three focus on probability and probability understanding from various angles. The focus is on early beginning and suitable contexts, which are often games. Contexts are very sensitive for learning paths in probability. The last two focus on statistical inference. Interestingly, the tight connection to probability induces automatically a wider perspective on inference. The constituents of risk, decision theory, and Bayesian inference shape the ideas and notions of statistical inference in an interplay rather than regarded in isolation.

Keywords:

Probabilistic thinking, Uncertainty; Decision theory; Subjectivist probability; Statistical inference

1. Introduction:

Probability is a virtual concept as Spiegelhalter (2014) has explained; Steinbring (1991) reinforces its theoretical character. A direct control of the "success" of probabilistic strategies is missing. Probability is situated at the crossing of several complementary relations: proportions and relative frequencies, expected value and relative frequencies, prediction and inference, probability and risk, weight of evidence and relative frequencies, etc. Yet, somehow, we have to shape probabilistic intuitions out of these relations.

There are several corner stones that may provide learning paths that lead to more sustainable probabilistic intuitions.

- First corner stone, 'Beginning early and developing the ideas and concepts spirally' letting the ideas grow and change in the learners' minds by suitable contexts and meaningful activities (as has been one insight of a special issue of Statistics Education Research Journal on "Building Future Generations of Statisticians").

- Second corner stone, 'Use games intelligently to let sustainable probabilistic

intuitions grow letting the young children have a direct access to shaping these ideas so that they recognise their purpose and intention right from the beginning.

- Third corner stone, 'Build thinking in Bayesian and risk terms as early as possible' to provide a wider spectrum of probabilistic literacy' letting the young learners play with probabilistic conclusions to experience their potential.

- Fourth corner stone, 'Connect probability with statistical inference right from the beginning' letting young learners experience a wide approach to probabilistic conclusions, which also provides a deeper understanding of probability.

- Fifth corner stone, 'Develop the twin relation between probability and risk to clarify the purpose of probability' bridging the gap between probability 'estimations' and decisions.

The first three cornerstones deal with early education that focuses on a wider meaning of probability than the current way of relating probability to relative frequencies. That should enable a spiral conceptual growth in learners that leads to sustainable conceptions. The last two cornerstones focus on the tight link between probability, statistical inference, and decision theory; they allow for a new evaluation of the question of Informal Inference as complete reduction of statistical inference to resampling techniques and simulation versus informal explorations of decision situations and methods of statistical inference.

First corner stone. Beginning early and developing the ideas and concepts spirally

Probabilistic intuition does not develop spontaneously, so it is necessary to train it and one should start from the initial level. Early education encourages the development of different verbal expressions, a didactic task that Varga (1983) was well aware and he invented several ways of body movements and hand signs to express a probability scale.

This early start of teaching probability and statistics also gives a chance to let the ideas grow and change in the learners' minds by suitable contexts and meaningful activities. One of the insights of a special issue of Statistics Education Research Journal on "Building Future Generations of Statisticians" (Borovcnik, Bilgin, & Howley 2020) was that it is essential to start educational interventions as early as possible (Borovcnik 2020): "Beginning early and developing the ideas and concepts spirally."

Piaget and Inhelder (1951) have shown by analysing their experiments that the intellectual growth follows a structure of stages so that certain tasks are uninteresting and unsolvable and the related concepts cannot be acquired. Yet, not only Todd and Gigerenzer (2000) argue that heuristics evolve if the language barrier is circumvented by suitable contexts, then the children can develop adequate strategies also at a stage when formally they would not have the means of understanding it. Thus, the task is to find such suitable contexts. Terán (2021) is a good example how successful teaching can be even at kindergarten level.

Second corner stone. Use games intelligently to induce sustainable probabilistic intuitions

Hirsh-Pasek and Golinkoff (2008, p. 3) state, "When children play they are learning." The objective is to base sustainable intuitions by letting the young children have a direct access to shaping these ideas so that they recognise their purpose and intention right from the beginning. An ideal source for that is games as children live in their own world were games play an essential part. Many games have their own and often complicated rules but the children develop an intuition what effect the rules have and how to optimise the decisions required in the game. The natural interest in the games, the context in which the concepts

are embedded bear their natural meaning, that all helps not only for the motivation of the children, as they are catalysts in shaping their ideas. Malaspina & Malaspina (2020) have gone beyond the simple game approach and ask the children to invent new rules for a game, which they have just played. That provides a boost to their efforts and a boost to their understanding of the concepts as a change of rules should influence the progress of the game in a certain, desirable way. The children swap their role from learner of mathematical concepts to constructors of mathematical concepts.

Third corner stone. Build thinking in Bayesian and risk terms as early as possible

From the conclusions of Martignon and Krauss (2009, p. 144), we underpin the ways of opening probabilistic concepts to children and that this endeavour can actually be successful:

> "The studies we present here investigate an approach to build a solid basis for the later acquisition of the calculus of probability. We propose the consistent use of materials like cards, tinker cubes, and tinker towers for training in basic inference and proportional thinking. Our first analyses indicate that proportional and probabilistic reasoning based on hands-on activities with these tools constitute a successful step towards probabilistic comparisons for decision-making and reckoning with risk. Our results encourage further experiments in that direction. It is promising to see that some of our collaborating teachers have quickly integrated our practices into their standard repertoire and develop their own units, with their own creative approaches encouraging children to work with tinker cubes and tinker towers."

The tools used in teaching provide a wider spectrum of probabilistic literacy letting the young learners play with probabilistic conclusions to experience their potential.

Fourth corner stone. Connect probability with statistical inference from early teaching

Probability without statistical inference is meaningless, statistical inference cannot be understood without a sound comprehension of probability. This insight changed the curricular endeavours in the mid-1980s when after a first round of introducing probability into school curricula, attempts followed to design learning paths towards statistical inference. It became immediately clear that statistical inference would widen the focus on probability interpretations. There were considerations to use resampling techniques and non-parametric statistics as an intermediate state for learning paths towards the full complexity of statistical inference (Borovcnik 1996, 2006).

The first attempts in the mid-1990s were doomed to failure as the computer capacity was insufficient for the wide application of resampling techniques. This changed after the Millennium and Cobb (2007) suggested replacing statistical inference completely by resampling techniques grounded on a pure and narrow frequentist concept of probability. This approach is now widely known as "Informal Inference". Borovcnik criticised this for various reasons, one of them the reduction of probability to a degenerate frequentist conception, another was that statistical inference involves more complex concepts such as type-II errors and that this would no more be expressible within a pure resampling framework. The critique is well covered in Borovcnik (2019), where also alternatives for a transient reduction towards meaningful procedures are discussed. Batanero and Borovcnik (2016) provide examples of informal explorations of concepts of statistical inference. Situations, contexts and decisions in which key concepts of statistical inference attain their intuitive meaning.

Borovcnik, Vancsó, and Fejes-Tóth (2020) try to continue this approach by connecting to ideas of Varga that have been expressed already in the 1970s when the didactic community still thought that statistical inference is too complex for teaching in the secondary school (Varga 1972, 1982, 1983).

It is remarkable that the informal explorations of statistical inference come repeatedly back to Bayesian ideas, decision-oriented probability and statistics, and a wider conception of probability embracing subjectivist and objectivist meanings (Martignon & Krauss 2009,

Vancsó 2006, 2009). Vancsó is insofar interesting, as his approach is to teach classical and Bayesian inference in parallel, which is seemingly opposed to Informal Inference (see also newer formulations and examples such as delMas 2017). By the way, Carranza and Kuzniak (2008) have analysed the negative impact of reducing probability to a purely frequentist notion on the students' perception of the methods to learn.

Fifth corner stone. Develop the twin relation between probability and risk

It is very helpful to develop a tight connection between probability and risk as is helps to clarify the purpose of probability' bridging the gap between probability 'estimations' and decisions. Probability can help to explore risky situations and make the final judgements more rational (Kapadia 2021).

> "The usual contract in games and in the insurance situation is signified by a swap of risks. One stakeholder who has no risk offers to the other to 'take over the risk' who then asks for monetary compensation. To fix the prices for such an exchange, it is essential to determine the probabilities of all (not only the adverse) outcomes – either by APT, FQT or SJT information and calculate an economic value of the diversity of outcomes."

The history of probability can be re-interpreted as a history of dealing with risk (Borovcnik 2016, Borovcnik & Kapadia 2014). This also explains that misconceptions in probability are robust against teaching interventions as risk is an even more multi-faceted concept than probability, see also the five definitions of risk of Hansson (2007), which all combine probability and risk.

If the concepts of probability and risk have emerged in a twin relation, then it gets plausible that probabilistic thinking is signified by a kind of archetypical forms of thinking (Batanero & Borovcnik 2016). Furthermore, risk brings probability close to decisions that have to be made under uncertainty, where the one-off decision requires different strategies and conceptions as are needed for decisions that are repeatedly done (or can at least be thought as if they were repeatable under the same circumstances). As a consequence, probability opens to diverse interpretations: probability embraces subjectivist ideas (SJT), frequentist (FQT) as well as equiprobability (APT; a priori theory, as probabilities are given a priori by counting the possibilities; for the terminology, see Borovcnik & Kapadia 2018).

Conclusions

It is essential to start teaching early. It is decisive to use suitable contexts that have a learning potential for young learners. The diverging dimensions of the concepts of probability and statistical inference speak for a diversity of approaches and a multi-faceted concept as objective for teaching interventions and learning paths. Reducing the concepts to make them more palatable would lead to biased understanding.

Yet, the ways, how to develop learning environments, especially for younger learners, but not only for them, have to be carefully designed. Furthermore, they have to be tried out in experimental teaching units that require a careful monitoring and investigation. Thirty years after Chance Encounters – Probability in Education (Kapadia & Borovcnik 1991), still a lot of work for research in the didactics of probability and statistical inference.

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