A hierarchical multidimensional graded response model with application to the study of sustainability perception

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Abstract

Hierarchical multidimensional item response theory (IRT) models have been developed to incorporate a general trait and more than one dimension of specific traits through different latent structures. In this study, we propose to insert a hierarchical multidimensional structure in the Samejima's graded response model. In order to estimate the parameters of the proposed model, we use a Bayesian approach through the No-U-Turn Sampler algorithm, a Markov chain Monte Carlo method. Simulation study is conducted to evaluate the parameter recovery in different scenarios. The results indicated that NUTS algorithm properly recovers all parameters and is accurate for all simulated scenarios. In order to demonstrate the practical applications of the proposed model, we consider actual data about sustainability perception of the Paraná III Basin residents in Brazil. The results suggest that (a) the model is appropriate to the data; (b) compared with the unidimensional graded response model, the proposed model better describe the data, providing useful information about the dimensions of specific traits and the relationship between them and the general trait; and (c) the use of a model with a hierarchical multidimensional structure should be based on the context of the data.

Keywords: Bayesian estimation; environmental sustainability; hierarchical MIRT model; latent variable modeling; multidimensional IRT.

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1 Introduction

Item Response Theory (IRT) models are part of statistical models that relate a set of observable variables to a set of unobservable ones, usually called latent variables or latent traits (Rao and Sinharay, 2007). They have been a mainstream class of statistical models since the 1970s for analyzing responses from assessment data. Situations where it is desired to measure multiple latent traits per individual or conditions where the latent features unidimensionality assumption is violated required IRT models with multidimensional latent structure, called multidimensional IRT (MIRT) models. Hierarchical models are included in the MIRT model class and consider a general latent trait associated at a different level to specific latent traits (Sheng and Wikle, 2008).

The Bayesian approach using Markov chain Monte Carlo (MCMC) was used to estimate the proposed model parameters through the No-U-Turn Sampler (NUTS) algorithm (Hoffman and Gelman, 2014), an extension of the Hamiltonian Monte Carlo (HMC) method (Neal, 2011). Simulation study indicated that NUTS algorithm properly recovers all parameters for all simulated scenarios.

In this study, we propose to insert a hierarchical multidimensional structure in the Samejima's graded response model, an IRT model used to score assessments and questionnaires composed of items with multiple ordered response categories. Specifically, the hierarchical multidimensional structure inserted in the model admits that each specific latent trait related to a respondent is a linear function of the general latent trait. The proposed model was applied to actual data about sustainability perception of the Paraná III Basin residents in Brazil.

2 Model formulation

The proposed hierarchical structure admits that each specific latent trait θ_{di} related to respondent i is a linear function of the general latent trait θ_{0i} such that

$$\theta_{di} = \gamma_d \theta_{0i} + \varepsilon_{di}, \tag{1}$$

where $\gamma_d > 0$ is a measure of association between the general latent trait and the dth specific latent trait, and $\varepsilon_{di} \sim N(0, 1)$.

Thus, we obtain the latent linear predictor of the model

$$\eta_{ijk} = b_{jk} - \sum_{d=1}^{D} q_{jd} a_{jd} (\gamma_d \theta_{0i} + \varepsilon_{di}). \tag{2}$$

The scoring's probability in a specific response category in the Samejima's model (Samejima, 1969) is modeled from the cumulative probabilities P^{+}_{ijk}, which

are of a respondent i to select in item j a specific response category k or another lower category, usually defined by the logit link function, given by the logistic cumulative distribution, that is,

$$P^{+}_{ijk} = P(Y_{ij} \leq k|\boldsymbol{\varphi}_{ij}) = \frac{\exp(\eta_{ijk})}{1+\exp(\eta_{ijk})}, \qquad k = 1, 2, \ldots, m_j - 1. \tag{3}$$

Thus, the probability of a respondent i to select a category k in item j is obtained by subtracting the adjacent cumulative probabilities, that is

$$P_{ijk} = P(Y_{ij} = k|\boldsymbol{\varphi}_{ij}) = \begin{cases} P^{+}_{ijk}, & \text{if } k = 1 \\ P^{+}_{ijk} - P^{+}_{ij[k-1]}, & \text{if } 2 \leq k < m_j. \\ 1 - P^{+}_{ij[k-1]}, & \text{if } k = m_j \end{cases} \tag{4}$$

3 Bayesian estimation

The likelihood function for proposed model is written as

$$\ell(\mathbf{Y}|\boldsymbol{\varphi}) = \prod_{i=1}^{n}\prod_{j=1}^{J} P(Y_{ij} = y_{ij}|\boldsymbol{\varphi}_{ij}). \tag{5}$$

Prior distributions are defined as follows: a_{jd} and γ_d have a truncated normal distribution, b_{jk} and θ_{0i} have a normal distribution. As the posterior distributions has an intractable form, NUTS algorithm was used via Stan software for parameter estimation.

4 Application

The research data correspond to part of the dataset of a case study conducted by Vincenzi et al. (2018) in Brazil. In this study, data from 420 individuals who responded to 26 items linked to the three specific sustainability dimensions: economic, environmental, and social were considered.

Proposed model was fit with 2,000 iterations, with the first 1,000 being discarded. The parameters' posterior distribution convergence was analyzed using the potential scale reduction factor's estimate, indicating convergence in all cases.

The questionnaire was validated based on the discrimination parameters estimates, discarding items with multidimensional discrimination (MDISC) lower than 0.45. Table 1 presents the validated items, MDISC values, and the mean difficulty b_{mean}.

It was also possible to observe a general discrimination power for each item, indicated by $MDISC$ values, ranged from 0.46 to 1.40. Items 1, 7, 10, 12, 13,

Table 1: Validated items and item parameters description.

Items	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
MDISC	.52	0.93	.71	.92	1.15	1.13	0.53	0.75	.62	0.57	1.04	.54	.54	.88	1.40	1.03	.56	.46	.50
b_{mean}	.14	1.86	.48	.14	-0.83	1.52	-1.45	-0.10	.17	-0.39	0.36	.01	.47	.32	-0.27	0.25	.29	.80	.94

17, and 18 had the lowest multidimensional discrimination, indicating that they did not discriminate well between individuals with high perception and those with low perception. On the other hand, items 5, 6, 11, 15, and 16 had the highest multidimensional discrimination, indicating that these items discriminate well the individuals. The mean difficulty values ranged from -0.83 to 1.86, indicating that these items allowed to assess the individuals' sustainability perception along the scale.

Figure 1 shows the general latent trait's density and the three specific latent trait's dimensions: economic, environmental, and social ones.

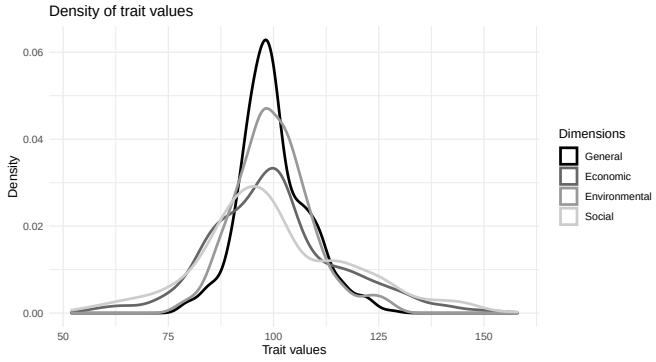

Figure 1: Density of general and specific latent traits.

The latent traits dimensions densities showed that the general latent trait had a lower dispersion than the specific latent traits dimensions. Furthermore, part of the individuals is probably likely to need intervention in the social area, as the chart shows a concentration of individuals with a social dimension value below expectations.

5 Final conclusions and remarks

We proposed to incorporate a hierarchical configuration to relate the general and specific latent traits in the Samejima's model. The structure hierarchical assumes that each specific latent trait depends on the general latent trait. Model

parameter estimation was performed using the NUTS algorithm, an alternative estimation method to traditional MCMC methods in the Bayesian approach.

The results presented in the application illustrate the use of the proposed model. In addition, the proposed model indicates that the general level of sustainability perception influenced the specific dimensions (economic, environmental, and social). Therefore, it is notable that the new model provides detailed information about the sustainability perception for the proposal of public policies.

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