# Population-Size-Dependent Multitype Models with Dependent Offspring: The Limiting Behaviour

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## Introduction

The limiting behaviour of Population-Size-Dependent Branching Processes (PSDBP) in discrete time has been widely studied in both simple and multitype case (see Klebaner (1985, 1989)). Furthermore, for the univariate case, dependent offspring has been allowed (see Cohn and Klebaner (1986)), but multidimensional version of this problem has not been studied yet.

In our paper, we have considered a Multitype PSDBP $\{Z(n)\}_{n\geq 0}$, with dependent offspring (see González et al. (2003)), where:

$$Z(0) \in \mathbb{N}_0^m, \quad Z(n+1) = (Z_1(n+1), \ldots, Z_m(n+1)) = \sum_{i=1}^{m} \sum_{j=1}^{Z_i(n)} X^{i,n,j}(Z(n))$$

($\sum_1^0 = 0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), being $m \in \mathbb{N}$ (the number of different types). For each $z \in \mathbb{N}_0^m$, $\{(X^{i,n,j}(z), i = 1, \ldots, m, j \in \mathbb{N})\}_{n\geq 0}$ is a sequence of nonnegative, integer-valued and identically distributed random vectors. Moreover, if $n, \overline{n} \in \mathbb{N}$ where $n \neq \overline{n}$, then $X^{i,n,j}(z)$ and $X^{k,\overline{n},l}(\overline{z})$ are independent for all $i, k = 1, \ldots, m$, $j, l \in \mathbb{N}$ and $z, \overline{z} \in \mathbb{N}_0^m$. Dependent offspring is allowed among types and brothers.

In this work, we study the limiting evolution for this model, suitably normalized, supposed that the extinction of the process is not an almost sure event. In order to guarantee that $P(\|Z(n)\| \to \infty) > 0$, we shall assume from now on the following hypotheses (see González et al. (2003)):

1. $P(Z(n) \to 0) + P(\|Z(n)\| \to \infty) = 1$, being $\|\cdot\|$ an arbitrary norm in $\mathbb{R}^m$.
2. There exits $m_{ij} = \lim_{\|z\|\to\infty} E\left[X_j^{i,0,1}(z)\right]$ for all $1 \leq i, j \leq m$ and $z \in \mathbb{N}_0^m$, and suppose $M = (m_{ij})_{1\leq i,j\leq m}$ an irreducible matrix with Perron-Frobenius eigenvalue $\rho > 1$.
3. $Var[Z(n+1)\mu | Z(n) = z] = O(z\mu)$, where $\mu$ is the right eigenvector associated to the Perron-Frobenius eigenvalue $\rho$.

In particular, we investigate conditions for the almost sure, $L^1$ and $L^2$ convergence of the sequence $\{Z(n)\mu/\rho^n\}_{n\geq 0}$ to a finite non-degenerate random variable and then we obtain conditions for the convergence of the sequence $\{Z(n)/\rho^n\}_{n\geq 0}$ to a non-degenerate random vector.

**Results**

Let

$$G_\alpha(z) = (E[|Z(n+1)\mu - E[Z(n+1)\mu|Z(n)=z]|^\alpha|Z(n)=z])^{1/\alpha}$$

for $\alpha = 1, 2$. Let also $M(z) = \left(E\left[X_j^{i,0,1}(z)\right]\right)_{1\le i,j\le m}$ and $D(z) = M(z) - M$ for all $z \in \mathbb{N}_0^m$. Now, we can establish the following results.

**Theorem 1.** *If $\{Z(n)\}_{n\ge 0}$ is a PSDBP with dependent offspring such that $\rho > 1$ and $G_2^2(z) = O(z\mu)$, then*

$$Z(n+1)\mu/Z(n)\mu \to \rho \text{ almost surely on the set } \{\|Z(n)\| \to \infty\}.$$

*Moreover, if there exits a non-increasing sequence $\{d(n)\}_{n\ge 1}$, with $\sum_{n=1}^{\infty} n^{-1}d(n) < \infty$ and such that $\|D(z)\| \le d(\sum_{i=1}^m z_i)$ for all non-null vector $z = (z_1, \dots, z_m) \in \mathbb{N}_0^m$, then $\{Z(n)\mu/\rho^n\}_{n\ge 0}$ converges almost surely to a finite random variable $W$ and $\lim_{n\to\infty} E[\rho^{-n}Z(n)\mu] > 0$.*

**Theorem 2.** *Suppose the conditions of theorem 1 hold and there exists a sequence $\{g_\alpha(n)\}_{n\ge 1}$, such that $\{g_\alpha(n)/n\}_{n\ge 1}$ is non-increasing, for $\alpha \in \{1, 2\}$. If it is verified:*

$$\sum_{n=1}^{\infty} n^{-2} g_\alpha(n) < \infty$$

*then $\{Z(n)\mu/\rho^n\}_{n\ge 0}$ converges in $L^\alpha$ to a finite non-degenerate random variable $W$. Moreover, if $M$ is a positively regular matrix, then $\{Z(n)/\rho^n\}$ converges in $L^\alpha$ and almost surely to $W\nu$, being $\nu$ the left eigenvector of $\rho$ such that $\nu\mu = 1$.*

## REFERENCES


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## RÉSUMÉ


*On considère un processus de branchement multitype et taille-dépendant avec dépendance de descendances. On a étudié la limite de processus normalisé. On a obtenu des conditions de convergence prèsque sûrement, convergence $L_1$ et $L_2$ vers un vecteur aléatoire non-dégénéré.*