The Statistical Work of Christian Huygens

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The statistical work of Huygens has a special place in his complete work. It consists of only 12 printed pages, as an appendix to someone else's book, and incidental remarks in some of his letters.

But in these 12 pages we find the origin of mathematical probability as a science. Just as he was the first to describe physical phenomena in mathematical formula's, he also was the first to apply what we nowadays call a model to problems of chance. His elegant reasoning, based on just 3 propositions, made it possible to solve all kind of problems related to games of chance. The technique has not changed since. We still employ the same methods for solving probability problems. What is most striking is that the text is so clear that it could serve even today for instructional purposes.

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With special thanks
to Mr. D. Berze for
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Introduction

The International Statistical Institute ISI is the world-wide society of professional statisticians. The Institute and its sections have some 4800 members from all disciplines of statistical science, pure as well as applied.

Several famous scientists are honoured by having their names given to sections, committees and medals. There is e.g. the Bernoulli Society for Mathematical Statistics and Probability, the Mercator Committee for Geographic Information Systems and the Quetelet medal for outstanding services to the ISI.

Christian Huygens is honoured as a pioneer of statistics by naming the ISI Committee for the History of Statistics after him. That the permanent office of the ISI is located at Voorburg should be regarded as pure chance however …

The ISI is pleased to contribute to the Huygens Memorial Festivities by means of a lecture on the statistical work of this truly international genius.

On ‘Van Rekeningh in Spelen van Geluck’

It was a strange sensation. I was standing at the desk of the old printed works department of the Royal Library at The Hague and I was handed a book that was more than 300 years old. Not being a professional historian, but just a simple statistician that was more familiar with modern computers, it was kind of a shock to have in my hands this book that would have been nearly forgotten, had it not contained that appendix of 12 pages. But this appendix was the first printed work on statistics and it would remain so for a long time. It’s influence has been remarkable. Still today in elementary courses on probability the same method of reasoning is used.

The basic technique of Huygens is that of repeatedly applying the first three propositions. By simply analyzing the problem and taking it apart in separate underlying and simpler problems, he is able to solve all the ‘gambling’ questions that were fashionable in his day and are more or less hated by present-day students who have to take statistics courses as an obligatory part of their studies.

In 1654, at the age of 26 Huygens visited Paris, but did not succeed in his plans to meet with Pascal and Fermat. However he heard of some gambling problems they were studying and also that they had found correct solutions for them. What seems remarkable is that they did not reveal their methods. This was customary in that time, as mathematicians jealously guarded their findings until a public contest. They often relied on such contests in order to secure their status and augment their income. So it was in their own interest to keep their ‘tools’ secret.

Back in The Hague, Huygens started to write a study about these problems. He not only reached the same results as Fermat and Pascal, but also solved some extended problems. The most innovating aspect of his work however is the clear exposition of his technique. He first states three simple propositions and then shows how the application of just these three propositions would work for all these problems. He wrote his results in the form of a letter to Franciscus Van Schooten, his former professor of Mathematics at Leiden University. In an accompanying letter, dated 27 April 1657, he suggested that Van Schooten could add this text to the book he was writing on mathematics. Van Schooten therefore translated the Dutch letter into Latin and published it under the Latin title ‘De Ratiociniis in Aleae Ludo’ at the end of his 1657 book ‘Excercitationes Mathematicarum’. Three years later, in 1660, the Van Schooten book was published in Dutch and Huygens’ text appeared under its original name ‘Van Rekeningh in Spelen van Geluck’.

The Introduction by Van Schooten and Huygens

At the end of his book, Van Schooten writes, in the ornate style of his time, ‘To the Reader’:

…Now that I have decided to conclude these exercises, I have found, my dear reader, that there were left several other things of amusing and excellent nature which, had I treated them according to their value and added them to these chapters, they would have increased its beauty, but possible also have brought usefulness and profit to your studies, but the trouble of describing them as well as the labour needed would have been vexatious to me…

He goes on to say that instead of these omitted subjects it seems not inappropriate to him to present his reader with the results the most honourable and far-famed gentleman Christian Huygens has achieved with computations in gambling games. He then sticks a feather in his own cap by stating that Huygens did find his results by applying Analysis, the foundations of which were taught to Huygens by himself.

After that the accompanying letter of Huygens follows. It starts by stating that he knows that Van Schooten is publishing his book just to illustrate how widely the potency of Algebra can be applied.

…so I do not doubt that what I have described in ‘Van Rekeningh in Spelen van Geluck’ will not be unfit to your goal…

Then Huygens more or less denies his priority rights:

…furthermore it should be known that already during some time, some of the most famous mathematicians of France have occupied themselves with this kind of computations, so that no one shall attribute the honour of having invented this first to me … but they all have kept their methods of invention veiled, so I needed everything to research and understand myself from the start. Therefore I am not sure if we base ourselves on the same basic principles. But in many problems I have found my solutions to be the same as theirs…

At the end, he says something on the relevancy of this work. He writes that at the end he has added some problems, without given their solution. He says he has done so because it would have given him too much work otherwise, but also because:
…it seemed advisable to me to leave something left that the readers (if there happen to be) could serve as exercise and pastime…

His doubts were unjust, as there have since been many readers and the work has had a tremendous influence on the further development of the statistical sciences.

The Basic Propositions 1-3

As with most scientific works of that time, Huygens presented his results in the form of ‘Propositions’. The first three of them form the entire basis upon which the rest of the propositions, which are problems to be solved, are based.

Proposition 1.

If I have equal chances to get a or b,
this is of a value \F(a + b,2) to me.

After stating such a proposition, Huygens provides an ingenious proof *and* an example with real numbers. Especially this last is important for the readers understanding.

Proposition 2.

If I have equal chances to get a or b or c,
this is of a value $\frac{a + b + c}{3}$ to me.

At the end of the proof of this proposition, Huygens generalizes the problem by writing: In the same way it is found that if I have equal chances on getting a or b or c or d, this is of a value $\frac{a + b + c + d}{4}$ to me, etc. Here again we see the hand of the genius, completely mastering the problem.

Proposition 3.

If p is the number of chances I have to get a and q is the number of chances I have to get b, supposing that every chance can occur evenly, it is of a value $\frac{pa + bq}{p + q}$ to me.

It is especially this third proposition that forms the basis for the solution of all the other problems.

The ‘Division Problem’ Propositions 4 - 9

Propositions 4 - 9 handle several varieties of the so called division problem. This problem arises when, for some reason, a gambling game cannot be finished and one wants to decide how to divide the prize.

Proposition 4 is the simplest. It handles the problem of how to divide a prize that is to be taken by the first player who wins 3 games. One of them has already won 2 games, the other only 1. Huygens says that in order to find a solution for this kind of problems, it is necessary to first start with the simplest. He emphasises that we should only give attention to the number of lacking games, for it does not make a difference if the number of games to be won was not 3 but instead 20, in which case the players would have already won 19 and 18 games respectively. Now let’s quote Huygens:

…To find what part is fair, consider what would happen if we continued the game. It is sure that if I would win the first game, the contest would be over and I would have the prize, let's name this ***a***. But if the other wins, we both would have equal chances, each lacking one game and therefore each being entitled to \f(1,2) ***a***. It is sure now that I have equal chances to win or loose the first game. Therefore I have equal chances to get ***a*** or \f(1,2) ***a***, what by the first proposition is as much as if I would have half of both, which makes \F(3,4) ***a***, leaving \F(1,4) ***a*** for my opponent…

The next problems are all of the same kind, and Huygens here masterly demonstrates his technique of refering back to previous solutions. Proposition 5 is a good example. It handles the problem of how to divide a prize with one player lacking 1 game, while the other has still 3 to go. He does not specify how many games were agreed upon, nor does he mention which kind of games are to be played. These questions do not matter to the problem: Huygens gives just the bare facts and thus avoids any possible confusion that could arise when people focus on the type of game, or the number of games to be won. Just concentrating on the essential: I miss 1 game, my opponent misses 3.

To illustrate such a possible confusion, I 'completed' this problem by adding that the two players had agreed that the prize-taker should have won 6 games. Nearly everyone I presented this problem to came up with a solution like: the first player has thus won 5 games already and the second 3, so a fair division should be 5:3. But suppose we had agreed on 8 won games instead of 6? Following the same line of reasoning, the players then had reached 7 and 5 games already, thus the division had to be 7:5 which is quite different!
Another intuitive solution is to divide in 3:1, which is also incorrect. The only correct solution is the one Huygens provides.

He brings back the problem is to the basic question: what are the chances for each player to win the prize if the set of games could be completed? Huygens way of reasoning is so elegant it deserves a full quotation:

…Let us consider in what situation we would be if I would win the first game, or if he would win. If I won, I would get the prize of a value ***a***, but if he won the first game, he would still lack 2 games against the one I lack. But then we would have the situation of the former proposition, in which it is shown that I should get \F(3,4) ***a*** and he \F(1,4) ***a***. So I have equal chances to have ***a*** or \F(3,4) ***a***. By proposition 1 this gives a value of \F(7,8) ***a*** …

Nowadays we probably would solve the problem in the following way: at most 3 more games are needed to complete the game. Whenever I win a game, I can take the prize, only if my opponent wins the next three games in a row can he take the prize. These 3 games form 8 possible combinations, of which 7 are won for me and only 1 for my opponent. Hence we should share the prize in 7:1 parts. But in this solution we miss the step-by-step reasoning and the elegant inductional methods of Huygens.

Huygens concludes these problems with expanding them to three players (Proposition 8) and more players in general (Proposition 9). He then ends with a very conveniently arranged table for the various combinations of lacking games in the case of three players.

The Dice Problem Propositions 10 - 14

This part of the manuscript begins with an explanation that with one fair dice one has 6 equal chances for each of the faces. And with two dice 36 equally likely throws etc. The induction method is used too, as Huygens explains that one can compute the number of different throws of an arbitrary large number of dice by multiplying the former number of throws by 6.

In dice games, there are two basically different kinds of betting. One is based on the total sum of the face values, the other on the individual values and knows such games as ‘the first player to throw 2 sixes with 5 dice wins’. These two game types give rise to different problems, but they all can be solved by logically analyzing the problem and reducing it to simpler problems.

Especially important with these problems is the so called sequence effect. That means that there is no difference in outcome between two dice (or coins etc.) thrown simultaneously and one single dice thrown twice. A direct effect of that is that with 2 dice we have not only the combination 3 + 4, but also 4 + 3 and both have equal chances. A consequence is that with 2 dice we have 5 different throws summing up to 6, but 6 ones that have 7 as a total. This knowledge is essential in many dice games.

It seems so simple now, but we have to realize that is was quite revolutionary in Huygens’ time. Take for example the throwing of two coins. No one today will have a problem to understand that there are four possible outcomes: HH, TH, HT and TT (H = Head, T = Tail). So the chance of throwing a Head with two coins, or with one coin thrown twice, is 3/4. But even the great French mathematician D’Alembert gave a value of 2/3, in an article in his 1754 Encyclopédie, some 100 years later! In his reasoning he did not distinguish between TH and HT, probably because the problem was stated as a simultaneous throw. Strangely enough, in modern physics this idea is not always true. Our daily-life idea of probability is synonymous with the so-called Maxwell-Boltzmann model of particles. Every particle has equal chances reaching any state, so the order TH is to be considered as different from the order HT. This applies for e.g. molecules in a gas.

But in the Bose-Einstein model particles are indistinguishable and there D'Alembert would have been right. In the race for proving this model, very recently an American research group published a positive result, reached at extremely low temperatures achieved by 'cooling' atoms by trapping and slowing them down with laser beams.
And to make life even more complicated there also is the Fermi-Dirac model for the orbits of electrons around a nucleus. The Pauli exclusion principle forbids electrons to have the same 'state'. Applied to our coins this would mean that when the first coin shows Head, the second coin only can turn Tail.

Let us return to Huygens. He precedes the dice problem propositions by a little table, giving the number of possible throws with 3 dice that sum up to 3 or 18, 4 or 17 etc.

I will not discuss in detail all these problems, but confine myself to Proposition 10, which handles the problem of the odds of throwing a six with 1 dice. It starts simple: one has 1 chance of winning ***a*** and 5 chances of winning 0. According to the 3rd Proposition (the book shows a misprint and mentions the 2nd Proposition) this has a value of \F(1 x ***a*** + 5 x 0,1 + 5) = \F(1,6) ***a***. Thus a bet on this should be based on a 1 to 5 stake. Then the problems continue with 2 dice. Again the induction principle: if one throws a six in the first turn, one wins ***a***. If one misses the first throw, then one can take a second throw, but then we have the former problem of throwing once and we just had computed that this has a value \F(1,6) ***a***. So the total value is computed from 1 chance on ***a*** and 5 chances on \F(1,6) ***a***. Applying the now familiar formula gives \F(11,36) ***a***. So the stakes for this bet should be 11 to 25.

The technique culminates in a very elegant handling of the problem of computing the chances for two players throwing with 2 dice, the one winning as soon as he throws a sum of 7, the other winning at throwing a 6 total. The player that has to throw

6 may start. The outcome is a chance of 30 to 31 for that player. This result is reached by writing the chances of the both players as the unknown values x and y and using this in our formula. This gives two equations which can be solved to get x and y. This use of algebra is new for these kinds of problems. Here again the book shows a misprint in giving the chance of the second player to be \F(31,62) instead of \F(31,61) .

The Last Propositions

After the 14 Propositions with their solutions, Huygens concludes the book with 5 Propositions, numbered 1 to 5, without a solution. As he already mentioned in his introductory letter, he left these to the interested reader for practice and pastime.

Number 1 is a variation on the problem in Proposition 14, now the first player throws once, then the second player twice and from then on both players twice. The winner is, as with the former problem the one who first throws a 6, respectively a 7. It seems that in Huygens' time there was an endless fantasy in designing games. The chances are 10355 to 12276.

Number 2 contains a novelty. Three players take a blind choice out of twelve disk, 4 of them being white, the other 8 being black. Whoever takes a white disk first wins. What are their chances? In fact, here the urn-model is introduced. Still today, many statistical problems are visualised by saying: take an urn with p white marbles and q black ones etc. For example, a technique to estimate the number of fish in a pond is to catch first say a hundred of them, marking them in an unobtrusive way and putting them back. Some time later again a bunch of fish are caught and from the ratio of unmarked to marked fish, the total population can be estimated. Few people will realise that Huygens was the first to introduce the basics of this model.

Number 4 uses the same 12 disks as in number 2. Here the question is about the chances to take 7 from them and have at least 3 white ones. This is a problem that (maybe with a slight variation) is still used today in statistics examinations, nearly 350 years after is was first formulated!

Number 3 deals with playing cards. What will be the chances of having one of each colour when we draw 4 cards out of a deck of 40, 10 from each colour. Huygens gives the (correct) answer of 1000 to 8139. Present-day bridge players still use Huygens' computational methods!

The last one again contains elements that are still actual in modern statistics. Two players start with 12 coins each, player A will give one to B when he throws a sum of 11 with 3 dice, player B will give one to A upon throwing a sum of 14. Winner is the player who first possesses all the coins. Many modern applications deal with this idea, e.g. what are the chances that one comes below a certain limit in a situation where gain and loss are randomly induced. In statistical quality control, a very up-to-date issue in statistics, the basic idea is that a product undergoes several consecutive stages of processing, and in each stage a random fluctuation can occur. These are modern applications of a more then 300 years old basic principle.

Huygens only gave the solutions for the problems 1, 3 and 5. Later Jacob Bernoulli in his 'Ars Conjectandi' published in 1713, 8 years after his death, would present answers to all these five problems, together with an explanation.

Conclusion

Huygens' statistical work, as published in the appendix to Van Schooten's book contains many modern elements. In his introductory letter he disclaims his priority rights, but from a certain point-of-view he is too modest in doing so. Especially his way of reasoning, his use of induction and his insight in what is essential to these kinds of problems is unprecedented, and his introduction of the urn model is undisputed.

Many of his problems are still in use in statistical education In fact his text could be used for this goal without any changes!

I have said here nothing about Huygens' other statistical interests. These can be found in the form of (parts of) letters to several scientists, both professional and amateur, although the distinction between these categories was neither strict nor important in the 17th century. In his correspondence with men like De Witt and Hudde, several problems related to life expectancy are discussed, problems which still have meaning for today's society.

Huygens statistical work is in a sense different from his other work in that it is not 'society driven'. With that term I mean the societal need for solutions that invokes scientific discoveries. Unless one wants to see the gambling problems of a certain upper class as problems of the society as a whole, only the discussions on life expectancy can be regarded as such.

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