Response Surface Methodology with Application in R

By

Dr. Ayubu Anapapa
Introduction

• *Response surface methodology*, or *RSM*, is a collection of mathematical and statistical techniques useful for the modelling and analysis of problems in which a response of interest is influenced by several variables and the objective is to optimize this response.

• For example, suppose that a chemical engineer wishes to find the levels of temperature \((x_1)\) and pressure \((x_2)\) that maximize the yield \((y)\) of a process.
• The process yield is a function of the levels of temperature and pressure, say 

\[ y = f(x_1, x_2) + \epsilon \]

where \( \epsilon \) represents the noise or error observed in the response \( y \).

• If we denote the expected response by \( E(y) = f(x_1, x_2) = \eta \), then the surface represented by \( \eta = f(x_1, x_2) \) is called a response surface.

• We usually represent the response surface graphically, such as in Figure 1, where \( \eta \) is plotted versus the levels of \( x_1 \) and \( x_2 \).
Figure 1: A three-dimensional response surface showing the expected yield ($\eta$) as a function of temperature ($x_1$) and pressure ($x_2$).
• To help visualize the shape of a response surface, we often plot the contours of the response surface as shown in Figure 2.

• In the contour plot, lines of constant response are drawn in the $x_1, x_2$ plane.

• Each contour corresponds to a particular height of the response surface.
Figure 2: A contour plot of a response surface
• The method of least squares is used to estimate the parameters in the approximating polynomials. Response surface analysis is then performed using the fitted surface.

• If the fitted surface is an adequate approximation of the true response function, then analysis of the fitted surface will be approximately equivalent to analysis of the actual system.

• The model parameters can be estimated most effectively if proper experimental designs are used to collect the data.

• Designs for fitting response surfaces are called response surface designs.
• RSM is a *sequential procedure*.

• Often, when we are at a point on the response surface that is remote from the optimum, such as the current operating conditions in Figure 3, there is little curvature in the system and the first order model will be appropriate.

• Our objective here is to lead the experimenter rapidly and efficiently along a path of improvement toward the general vicinity of the optimum.
Figure 3: The sequential nature of RSM
• Once the region of the optimum has been found, a more elaborate model, such as the second order model, may be employed, and an analysis may be performed to locate the optimum.

• The eventual objective of RSM is to determine the optimum operating conditions for the system or to determine a region of the factor space in which operating requirements are satisfied.

• More extensive presentations of RSM are in Khuri and Cornell (1996), Myers, Montgomery and Anderson-Cook (2016), and Box and Draper (2007). The review paper by Myers et al. (2004) is also a useful reference.
The Method of Steepest Ascent

• Frequently, the initial estimate of the optimum operating conditions for the system will be far from the actual optimum.

• The objective of the experimenter is to move rapidly to the general vicinity of the optimum.

• The method of steepest ascent is a procedure for moving sequentially in the direction of the maximum increase in the response.

• If minimization is desired, then we call this technique the method of steepest descent.
• The fitted first order model is

\[ \hat{y} = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i \]

• The first-order response surface (the contours of \( \hat{y} \)), is a series of parallel lines as shown in Figure 4.

• The direction of steepest ascent is the direction in which \( \hat{y} \) increases most rapidly.

• This direction is normal to the fitted response surface. We usually take as the path of steepest ascent the line through the centre of the region of interest and normal to the fitted surface.
Figure 4: First order response surface and path of steepest ascent
• Experiments are conducted along the path of steepest ascent until no further increase in response is observed.

• Then a new first order model may be fit, a new path of steepest ascent determined, and the procedure continued.

• Eventually, the experimenter will arrive in the vicinity of the optimum.

• This is usually indicated by lack of fit of a first order model.

• At that time, additional experiments are conducted to obtain a more precise estimate of the optimum.
Example 1

• A chemical engineer is interested in determining the operating conditions that maximize the yield of a process.

• Two controllable variables influence process yield: reaction time and reaction temperature. The engineer is currently operating the process with a reaction time of 35 minutes and a temperature of 155°F, which result in yields of around 40 percent.

• Because it is unlikely that this region contains the optimum, she fits a first order model and applies the method of steepest ascent.

• The engineer decides that the region of exploration for fitting the first order model should be (30, 40) minutes of reaction time and (150, 160) Fahrenheit.
Table 1: Process Data for Fitting the First Order Model

<table>
<thead>
<tr>
<th>Natural Variables</th>
<th>Coded Variables</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1$</td>
<td>$\xi_2$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>30</td>
<td>150</td>
<td>-1</td>
</tr>
<tr>
<td>30</td>
<td>160</td>
<td>-1</td>
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<tr>
<td>40</td>
<td>150</td>
<td>1</td>
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<tr>
<td>40</td>
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<td>1</td>
</tr>
<tr>
<td>35</td>
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<td>0</td>
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<tr>
<td>35</td>
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<tr>
<td>35</td>
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</tr>
<tr>
<td>35</td>
<td>155</td>
<td>0</td>
</tr>
<tr>
<td>35</td>
<td>155</td>
<td>0</td>
</tr>
</tbody>
</table>
• Note that the design used to collect these data is a $2^2$ factorial augmented by five centre points.

• This allows one to estimate the experimental error and to check the adequacy of the first-order model. A first order model may be fit to these data by least squares.

• This can be done in R as follows:

```r
x1<-c(-1,-1,1,1,rep(0,5))
x2<-c(-1,1,-1,1,rep(0,5))
y<-c(39.3,40.0,40.9,41.5,40.3,40.5,40.7,40.2,40.6)
library(rsm)
Ex1<-rsm(y~FO(x1,x2))
summary(Ex1)
```

• The results are shown in Figure 5
Estimate Std. Error  t value Pr(>|t|)
(Intercept)  40.444444  0.057288 705.9869  5.451e-16 ***
x1           0.775000  0.085932   9.0188  0.000104 ***
x2           0.325000  0.085932   3.7821  0.009158 **

---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Multiple R-squared:  0.941,    Adjusted R-squared:  0.9213
F-statistic: 47.82 on 2 and 6 DF,  p-value: 0.0002057

Analysis of Variance Table

Response: y

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FO(x1, x2)</td>
<td>2 2.82500</td>
<td>1.41250</td>
<td>47.8213</td>
<td>0.0002057</td>
</tr>
<tr>
<td>Residuals</td>
<td>6 0.17722</td>
<td>0.02954</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lack of fit</td>
<td>2 0.00522</td>
<td>0.00261</td>
<td>0.0607</td>
<td>0.9419341</td>
</tr>
<tr>
<td>Pure error</td>
<td>4 0.17200</td>
<td>0.04300</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 5: The Fitted First Order Model in Example 1**
• The fitted model in the coded variables is:

\[ \hat{y} = 40.44 + 0.775x_1 + 0.325x_2 \]

• The F-test for the overall regression is significant.

• We also have no reason to question the adequacy of the first order model.

• To move away from the design centre—the point \((x_1 = 0, x_2 = 0)\)—along the path of steepest ascent, we would move 0.775 units in the \(x_1\) direction for every 0.325 units in the \(x_2\) direction.

• Thus, the path of steepest ascent passes through the point \((x_1 = 0, x_2 = 0)\) and has a slope \(0.325/0.775\).
• The engineer decides to use 5 minutes of reaction time as the basic step size.

• Using the relationship between $\xi_1$ and $x_1$, we see that 5 minutes of reaction time is equivalent to a step in the coded variable $x_1$ of $\Delta x_1 = 1$.

• Therefore, the steps along the path of steepest ascent are $\Delta x_1 = 1.0000$ and $\Delta x_2 = (0.325 / 0.775) = 0.42$.

• The engineer computes points along this path and observes the yields at these points until a decrease in response is noted.

• The results are shown in Table 2 in both coded and natural variables.
Table 2: Steepest Ascent Experiment for Example 1

<table>
<thead>
<tr>
<th>Steps</th>
<th>Coded Variables</th>
<th>Natural Variables</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(x_1)</td>
<td>(x_2)</td>
<td>(\xi_1)</td>
</tr>
<tr>
<td>Origin</td>
<td>0</td>
<td>0</td>
<td>35</td>
</tr>
<tr>
<td>(\Delta)</td>
<td>1.00</td>
<td>0.42</td>
<td>5</td>
</tr>
<tr>
<td>Origin + (\Delta)</td>
<td>1.00</td>
<td>0.42</td>
<td>40</td>
</tr>
<tr>
<td>Origin + 2(\Delta)</td>
<td>2.00</td>
<td>0.84</td>
<td>45</td>
</tr>
<tr>
<td>Origin + 3(\Delta)</td>
<td>3.00</td>
<td>1.26</td>
<td>50</td>
</tr>
<tr>
<td>Origin + 4(\Delta)</td>
<td>4.00</td>
<td>1.68</td>
<td>55</td>
</tr>
<tr>
<td>Origin + 5(\Delta)</td>
<td>5.00</td>
<td>2.10</td>
<td>60</td>
</tr>
<tr>
<td>Origin + 6(\Delta)</td>
<td>6.00</td>
<td>2.52</td>
<td>65</td>
</tr>
<tr>
<td>Origin + 7(\Delta)</td>
<td>7.00</td>
<td>2.94</td>
<td>70</td>
</tr>
<tr>
<td>Origin + 8(\Delta)</td>
<td>8.00</td>
<td>3.36</td>
<td>75</td>
</tr>
<tr>
<td>Origin + 9(\Delta)</td>
<td>9.00</td>
<td>3.78</td>
<td>80</td>
</tr>
<tr>
<td>Origin + 10(\Delta)</td>
<td>10.00</td>
<td>4.20</td>
<td>85</td>
</tr>
<tr>
<td>Origin + 11(\Delta)</td>
<td>11.00</td>
<td>4.62</td>
<td>90</td>
</tr>
<tr>
<td>Origin + 12(\Delta)</td>
<td>12.00</td>
<td>5.04</td>
<td>95</td>
</tr>
</tbody>
</table>
• Increases in response are observed through the tenth step; however, all steps beyond this point result in a decrease in yield.

• Therefore, another first-order model should be fit in the general vicinity of the point \((\xi_1 = 85, \xi_2 = 175)\).

• A new first order model is fit around the point \((\xi_1 = 85, \xi_2 = 175)\). The region of exploration for \(\xi_1\) is \([80, 90]\), and it is \([170, 180]\) for \(\xi_2\).

• Thus, the coded variables are

\[
x_1 = \frac{\xi_1 - 85}{5} \quad \text{and} \quad x_2 = \frac{\xi_2 - 175}{5}
\]

• Once again, a \(2^2\) design with five centre points is used.

• The experimental design is shown in Table 3.
### Table 3: Data for Second First Order Model for Example 1

<table>
<thead>
<tr>
<th>Natural Variables</th>
<th>Coded Variables</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_1$</td>
<td>$\xi_2$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>80</td>
<td>170</td>
<td>-1</td>
</tr>
<tr>
<td>80</td>
<td>180</td>
<td>-1</td>
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<tr>
<td>90</td>
<td>170</td>
<td>1</td>
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<tr>
<td>90</td>
<td>180</td>
<td>1</td>
</tr>
<tr>
<td>85</td>
<td>175</td>
<td>0</td>
</tr>
<tr>
<td>85</td>
<td>175</td>
<td>0</td>
</tr>
<tr>
<td>85</td>
<td>175</td>
<td>0</td>
</tr>
<tr>
<td>85</td>
<td>175</td>
<td>0</td>
</tr>
<tr>
<td>85</td>
<td>175</td>
<td>0</td>
</tr>
</tbody>
</table>
Another first order model is fitted to the coded variables in Table 3 using R as follows:

\[ y_1 = c(76.5, 77.0, 78.0, 79.5, 79.9, 80.3, 80.0, 79.7, 79.8) \]

\[ Ex1b = \text{rsm}(y1 \sim FO(x1, x2)) \]

`summary(Ex1b)`

Results are displayed in Figure 6.

The fitted model is:

\[ \hat{y} = 78.97 + 1.00x_1 + 0.50x_2 \]
Estimate Std. Error t value Pr(>|t|)
(Intercept) 78.96667 0.45379 174.0156 2.43e-12 ***
x1 1.00000 0.68069 1.4691 0.1922
x2 0.50000 0.68069 0.7346 0.4903
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Multiple R-squared:  0.3102,  Adjusted R-squared:  0.08023
F-statistic: 1.349 on 2 and 6 DF,  p-value: 0.3283

Analysis of Variance Table

Response: y1

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FO(x1, x2)</td>
<td>2 5.000</td>
<td>2.5000</td>
<td>1.3489</td>
<td>0.3282610</td>
</tr>
<tr>
<td>Residuals</td>
<td>6 11.120</td>
<td>1.8533</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lack of fit</td>
<td>2 10.908</td>
<td>5.4540</td>
<td>102.9057</td>
<td>0.0003635</td>
</tr>
<tr>
<td>Pure error</td>
<td>4 0.212</td>
<td>0.0530</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 6: The Fitted Second First Order Model in Example 1
• The results indicate that the first order model is not an adequate approximation.

• This curvature in the true surface may indicate that we are near the optimum.

• At this point, additional analysis must be done to locate the optimum more precisely.
Analysis of a Second Order Response Surface

• When the experimenter is relatively close to the optimum, a model that incorporates curvature is usually required to approximate the response.

• In most cases, the second-order model is adequate;

\[ y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i<j} \beta_{ij} x_i x_j + \epsilon \]

• The fitted model can be used to find the optimum set of operating conditions for the x’s and to characterize the nature of the response surface.
Location of the Stationary Point

• Suppose we wish to find the levels of $x_1, x_2, \ldots, x_k$ that optimize the predicted response.

• This point, if it exists, will be the set of $x_1, x_2, \ldots, x_k$ for which the partial derivatives $\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = \cdots = \frac{\partial y}{\partial x_k} = 0$.

• This point, say $x_{1,s}, x_{2,s}, \ldots, x_{k,s}$, is called the stationary point.

• The stationary point could represent a point of maximum response, a point of minimum response, or a saddle point.

• These three possibilities are shown in Figures 7, 8 and 9.
Figure 7: Response surface and contour plot illustrating a surface with a maximum
Figure 8: Response surface and contour plot illustrating a surface with a minimum
Figure 9: Response surface and contour plot illustrating a surface with a saddle point
• Contour plots play a very important role in the study of the response surface.

• By generating contour plots using computer software for response surface analysis, the experimenter can usually characterize the shape of the surface and locate the optimum with reasonable precision.

• We may obtain a general mathematical solution for the location of the stationary point.

• Writing the fitted second-order model in matrix notation, we have
\[
\hat{y} = \hat{\beta}_0 + x'b + x'Bx
\]

where

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_k
\end{bmatrix}, \quad b = \begin{bmatrix}
\hat{\beta}_1 \\
\hat{\beta}_2 \\
\vdots \\
\hat{\beta}_k
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
\hat{\beta}_{11}, & \hat{\beta}_{12}/2, & \ldots, & \hat{\beta}_{1k}/2 \\
\hat{\beta}_{21}, & \hat{\beta}_{22}, & \ldots, & \hat{\beta}_{2k}/2 \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\beta}_{k1}, & \hat{\beta}_{k2}, & \ldots, & \hat{\beta}_{kk}
\end{bmatrix}_{\text{sym.}}
\]

- That is, \( b \) is a \((k \times 1)\) vector of the first-order regression coefficients.

- \( B \) is a \((k \times k)\) symmetric matrix whose main diagonal elements are the pure quadratic coefficients \((\hat{\beta}_{ii})\) and whose off-diagonal elements are one half the mixed quadratic coefficients \((\hat{\beta}_{ij}, i \neq j)\).
• The derivative of $\hat{y}$ with respect to the elements of the vector $x$ equated to 0 is

$$\frac{\partial \hat{y}}{\partial x} = b + 2Bx = 0$$

• The stationary point is the solution to the equation, or

$$x_s = -\frac{1}{2}B^{-1}b$$

• Substituting this into $\hat{y}$, we find the predicted response at the stationary point as

$$\hat{y}_s = \hat{\beta}_0 + \frac{1}{2}x'_s b$$
Characterizing the Response Surface

• Once we have found the stationary point, it is usually necessary to characterize the response surface in the immediate vicinity of this point.

• By characterize, we mean determining whether the stationary point is a point of maximum or minimum response or a saddle point.

• We also usually want to study the relative sensitivity of the response to the variables $x_1, x_2, \ldots, x_k$. 

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• This can be achieved by performing *canonical analysis*.

• The model is transformed into a new coordinate system with the origin at the stationary point $x_s$.

• The axes of this system are the rotated until they are parallel to the principal axes of the fitted response surface.

• This results in the *canonical form* of the fitted model

\[
\hat{y} = \hat{y}_s + \lambda_1 w_1^2 + \lambda_2 w_2^2 + \cdots + \lambda_k w_k^2
\]
• The \{w_i\} are the transformed independent variables and the \{\lambda_i\} are the eigenvalues or characteristic roots of the matrix B

• The nature of the response surface can be determined from the stationary point and the signs and magnitudes of the \{\lambda_i\}.

• Suppose that the stationary point is within the region of exploration for fitting the second order model;
  - If the \{\lambda_i\} are all positive, \(x_s\) is a point of minimum response.
  - If the \{\lambda_i\} are all negative, \(x_s\) is a point of maximum response.
  - If the \{\lambda_i\} have different signs, \(x_s\) is a saddle point.

• Furthermore, the surface is steepest in the \(w_i\) direction for which \(|\lambda_i|\) is the greatest.
Example 2

• We will continue the analysis of the chemical process in Example 1.

• A second order model in the variables $x_1$ and $x_2$ cannot be fit using the design in Table 3.

• The experimenter decides to augment this design with enough points to fit a second-order model.

• She obtains four observations at $(x_1 = 0, x_2 = \pm1.414)$ and $(x_1 = \pm1.4140, x_2 = 0)$.

• The complete experiment is shown in Table 4.

• This design is called a central composite design (or CCD).
<table>
<thead>
<tr>
<th>Natural Variables</th>
<th>Coded Variables</th>
<th>Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_1 )</td>
<td>( \xi_2 )</td>
<td>( x_1 )</td>
</tr>
<tr>
<td>80</td>
<td>170</td>
<td>-1</td>
</tr>
<tr>
<td>80</td>
<td>180</td>
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<td>90</td>
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<tr>
<td>85</td>
<td>167.93</td>
<td>0</td>
</tr>
</tbody>
</table>
• We will focus on fitting a quadratic model to the yield response $y$.
• R will be used to fit a response surface and to construct the contour plots.

```r
x1<-c(-1,-1,1,1,rep(0,5),1.414,-1.414,0,0)
x2<-c(-1,1,-1,1,rep(0,5),0,0,1.414,-1.414)
y<-c(76.5,77.0,78.0,79.5,79.9,80.3,80.0,79.7,79.8,78.4,
75.6,78.5,77.0)
library(rsm)
Ex2<-rsm(y~SO(x1,x2))
summary(Ex2)
```

• The results are shown in Figures 10 and 11
| Estimate | Std. Error | t value | Pr(>|t|) |
|----------|------------|---------|----------|
| (Intercept) | 79.939955 | 0.119089 | 671.2644 < 2.2e-16 *** |
| x1 | 0.995050 | 0.094155 | 10.5682 1.484e-05 *** |
| x2 | 0.515203 | 0.094155 | 5.4719 0.000934 *** |
| x1:x2 | 0.250000 | 0.133145 | 1.8777 0.102519 |
| x1^2 | -1.376449 | 0.100984 | -13.6303 2.693e-06 *** |
| x2^2 | -1.001336 | 0.100984 | -9.9158 2.262e-05 *** |

---

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Multiple R-squared: 0.9827, Adjusted R-squared: 0.9704
F-statistic: 79.67 on 5 and 7 DF, p-value: 5.147e-06

Analysis of Variance Table

Response: y

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FO(x1, x2)</td>
<td>2</td>
<td>10.0430</td>
<td>5.0215</td>
<td>70.8143 2.267e-05</td>
</tr>
<tr>
<td>TWI(x1, x2)</td>
<td>1</td>
<td>0.2500</td>
<td>0.2500</td>
<td>3.5256 0.1025</td>
</tr>
<tr>
<td>PQ(x1, x2)</td>
<td>2</td>
<td>17.9537</td>
<td>8.9769</td>
<td>126.5944 3.194e-06</td>
</tr>
<tr>
<td>Residuals</td>
<td>7</td>
<td>0.4964</td>
<td>0.0709</td>
<td></td>
</tr>
<tr>
<td>Lack of fit</td>
<td>3</td>
<td>0.2844</td>
<td>0.0948</td>
<td>1.7885 0.2886</td>
</tr>
<tr>
<td>Pure error</td>
<td>4</td>
<td>0.2120</td>
<td>0.0530</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 10: The Fitted Second Order Model of Example 2**
Stationary point of response surface:

\[
x_1 \quad x_2 \\
0.3892304 \quad 0.3058466
\]

Eigenanalysis:
eigen() decomposition
$\text{values}$

\[
[1] \quad -0.9634986 \quad -1.4142867
\]

$\text{vectors}$

\[
\begin{bmatrix}
x_1 & -0.2897174 & -0.9571122 \\
x_2 & -0.9571122 & 0.2897174
\end{bmatrix}
\]

\textit{Figure 11: Analysis of the Second Order Model of Example 2}
• The fitted model is
\[ \hat{y} = 79.94 + 0.995x_1 + 0.515x_2 - 1.376x_1^2 - 1.001x_2^2 + 0.250x_1x_2 \]

• The stationary point is \( x_{1,s} = 0.389 \) and \( x_{2,s} = 0.306 \). In terms of the natural variables, the stationary point is:

\[
\begin{align*}
0.389 &= \frac{\xi_1 - 85}{5} \\
0.306 &= \frac{\xi_1 - 175}{5}
\end{align*}
\]

which yields \( \xi_1 = 86.95 \approx 87 \) minutes of reaction time and \( \xi_2 = 176.53 \approx 176.5^\circ \text{F} \).

• Using \( \hat{y}_s = b_0 + \frac{1}{2} x'_s \hat{b} \), we may find the predicted response at the stationary point as \( \hat{y}_s = 80.21 \).
• The eigenvalues of \( B \) are \( \lambda_1 = -0.9634 \) and \( \lambda_2 = -1.4141 \).

• Thus, the canonical form of the fitted model is

\[
\hat{y} = 80.21 - 0.9634w_1^2 - 1.4141w_2^2
\]

• Because both \( \lambda_1 \) and \( \lambda_2 \) are negative, we conclude that the stationary point is a maximum.

• Contour and surface plots shown in Figures 12 and 13 are generated in R as follows
Figure 12: Contour Plot for the Model of Example 2
Figure 13: Response Surface Plot for the Model of Example 2
Examining the contour and surface plots shows that the optimum is very near 175°F and 85 minutes of reaction time.

The response is at a maximum at this point.

From examination of the contour plot, we note that the process may be slightly more sensitive to changes in reaction time than to changes in temperature.
Conclusion

• RSM is a sequential procedure.

• Start with low order polynomial models.

• Move rapidly and efficiently towards optimum region.

• Fit higher order models around the vicinity of the optimal region.

• Analyze the response surface fitted in the optimal region.
Thank You